Path Integrals and Renormalization

Contents

1 Euclidean Time and Field Theory: Path Integrals

Path integrals are an alternate/complementary representation of standard quantum theory; it is a different quantization procedure.

1.1 Quantum Theory in Euclidean Time

Consider a 1D non-relativistic particle of mass m in some external potential $V(q)$. Then, the Hamiltonian is classically given by:

$$\mathcal{H} = \frac{p^2}{2m} + V(q)$$

In the quantum theory, we associate a state $|\psi\rangle$ for the system, and we have position eigenkets $\{|q\rangle\}$. Suppose the particle is localised at some q, we can obtain this information by using the 'position operator' and acting it on the state:

$$Q\,|q\rangle = q\,|q\rangle$$

which is an eigenvalue equation.

The evolution of the quantum state is given by the time evolution operator $\mathcal{U}(t)$. This is another way to solve the Schrodinger equation. When the Hamiltonian is time independent (as it is now), then

$$\mathcal{U}(t) = \exp\left(\frac{-it}{\hbar}\mathcal{H}\right) \tag{1}$$

Then, if at $t = 0$ the state is $|\psi\rangle$, then at $t > 0$, the state is given by $|\psi(t)\rangle = \mathcal{U}(t)\,|\psi\rangle$.
In the language of path integrals we talk about the system in terms of probability amplitudes. If we have a possible trajectory between $(t_1, q_1), (t_2, q_2)$, the transition amplitude from the particle to travel between the two points is given by

$$K\left(q_2, t_2; q_1, t_1\right) = \langle q_2|\,\mathcal{U}(t_2 - t_1)\,|q_1\rangle = \int \mathcal{D}[q(s)]\,\exp\left(\frac{i}{\hbar}\mathcal{S}[q]\right) \tag{2}$$

with the boundary conditions $q(t_1) = q_1, q(t_2) = q_2$ (s is just a parameter for time). Importantly the $\mathcal{S}$ is the classical action, given by:

$$\mathcal{S}[q] = \int_{t_1}^{t_2} ds\left[\frac{m}{2}\left(\frac{dq}{ds}\right)^2 - V(q(s))\right] \tag{3}$$

Rigorously this can be shown by discretizing the time interval, adding the contribution from each interval and then taking the continuum limit.

Now, consider the same system at a finite temperature. This means that it is in equilibrium with some big thermal bath. In this situation, the particle can not be in a pure state, since it is entangled with the thermal ensemble. Hence, as opposed to asking what the state of the particle is, we ask what is the probability of getting a specific result, since now we only have statistical descriptions available. Such mixed thermal states are described by density matrices ρ. Specifically, the 'Gibbs state' (canonical ensemble) is given by:

$$\rho = \frac{1}{Z} \exp\left(-\beta \mathcal{H}\right) \tag{4}$$

where $Z = \mathrm{tr}\left(\exp\left(-\beta\mathcal{H}\right)\right)$ is the partition function, and $\beta = \left(k_B T\right)^{-1}$. The definition of Z implies that $\mathrm{tr}(\rho) = 1$. That is, for some observable A, we will have:

$$\langle A \rangle_T = \mathrm{tr}\left(\rho A\right) \tag{5}$$

The key idea is that the density operator (except for the normalization Z), looks a lot like an evolution operator at some "imaginary time" τ. This is nothing more than a formal mathematical trick. That is:

$$\exp(-\beta\mathcal{H}) = \mathcal{U}(-i\tau)$$
$$\implies \beta = \frac{\tau}{\hbar}$$

What is important to note is that the evolution operator is only defined when $\mathrm{Im}(t) < 0$. This has to do with the fact that when we expand the evolution operator (in real time) in the energy eigenkets, the sum is only convergent when $\mathrm{Im}(t) < 0$.

We now ask: Do we have a path integral representation for this object $\mathcal{U}(-i\tau) := \mathcal{U}_E(\tau)$, where the E stands for Euclidean. We claim that the path integral can be considered by doing the same discretization in Euclidean time, as $\Delta t = -i\Delta\tau$. Using this, we can define the Euclidean amplitude:

$$K_E(q_2, \tau_2; q_1, \tau_1) = \langle q_2 | \mathcal{U}_E(\tau_2 - \tau_1) | q_1 \rangle = \int \mathcal{D}[q(\sigma)] \exp\left(\frac{-\mathcal{S}_E[q]}{\hbar}\right)$$

where

$$\mathcal{S}_E = \int_{\tau_1}^{\tau_2} d\sigma \left[\frac{m}{2}\left(\frac{dq}{d\sigma}\right) + V(q)\right]$$

is the "Euclidean Action" associated to a history in Eucldiean time σ and is derived by taking $s \to -i\sigma$.

To look at mixed states now, we have to also compute the partition function Z which can be considered as the partition function of a quantum system or as a statistical average over the histories $q(\sigma)$. Why? Firstly, note that the new $\mathcal{S}_E$ is real and positive and it no longer a pure phase. Thus, this is more like the Boltzmann weight we have from classical statistical mechanics. Now we use the definition of the partition function:

$$Z = \mathrm{tr}\left(\mathcal{U}_E\left(\beta\hbar\right)\right) = \sum_n \langle n | \mathcal{U}_E\left(\beta\hbar\right) | n \rangle = \sum_n e^{-\beta E_n}$$

However, this can be rewritten (using the completeness of the position eigenkets) as:

$$Z = \int dq \, \langle q | \mathcal{U}_E(\beta\hbar) | q \rangle = \int \mathcal{D}[q(\sigma)] \exp\left(-\frac{\mathcal{S}_E(q)}{\hbar}\right)$$

where now the boundary conditions are periodic i.e. $q(\beta\hbar) = q(0)$ so the path integral is over periodic trajectories in Euclidean time.

(Insert the explanation from the Heisenberg picture operator)
The Euclidean period is:

$$\tau_\beta = \beta\hbar = \frac{\hbar}{k_B T} \tag{6}$$

Therefore, we have an equivalence between finite temperature and periodicity in Euclidean time. That is, we have the following dictionary:

$$\begin{aligned} \text{QM at finite temp} &\longleftrightarrow \text{Classical statistical trajectories in } \tau \\ \hbar &\longleftrightarrow \text{Temperature } T_{\text{stat}} \\ \text{Temp of quantum system } T_Q &\longleftrightarrow \text{Period } \tau_\beta \end{aligned}$$

2 Operators and Correlation Functions in Path Integrals

2.1 Operators and Observables via Path Integrals

To understand how these are described, we go back to the differences between the Schrodinger and the Heisenberg pictures of quantum theory, which are related by a change of basis (see [1]). Most importantly, recall that in the Schrodinger picture, state kets evolve with time, while base kets and operators are independent of time. However, in the Heisenberg picture, operators and base kets evolve in time, while the state kets are time independent. For time independent Hamiltonians $\mathcal{H}$, the time evolution operator is given by:

$$\mathcal{U}(t) = \exp\left(-\frac{it}{\hbar}\mathcal{H}\right)$$

Now, in the Heisenberg picture, the base kets at some time t, are given by:

$$\begin{aligned} |a(t)\rangle_H &= \mathcal{U}^\dagger(t)\,|a\rangle_S \\ \mathcal{A}_H &= \mathcal{U}^\dagger(t)\mathcal{A}_S\mathcal{U}(t) \end{aligned}$$

Aside: In classical physics as well, we sometimes do encounter two different descriptions of time evolution which contain the same physics. In fluid dynamics, we have the Eulerian and Lagrangian approaches to describe fluid flow, which are analogous to the present case.

In the context of path integrals, under the Heisenberg picture, we see that the probability amplitude for our system to go from (t_1, q_1) to (t_2, q_2) i.e. the propagator, can be rewritten as:

$$K(q_2, t_2; q_1, t_1) =_S \langle q_2|\,\mathcal{U}(t_2 - t_1)\,|q_1\rangle_S =_H \langle q_2, t_2|q_1, t_1\rangle_H$$

This stil gives us the same physics. However, we now want to know what happens to the path integral

$$\mathcal{I} = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}(q)\, \exp\left(\frac{i}{\hbar}\mathcal{S}(q)\right) q(t_1)$$

where $t_i < t_1 < t_f$. Note that this is simply a classical measurement, where we compute this path integral while we measure the position of the particle at some intermediate time t_1. We can do this by decomposing the above integral into two functional integrals. With the notation $q(t_1) = q_1$ we first look at all paths between q_i, q_1 and then at all paths between q_1, q_f, and then integrate over all possible q_1.
That is,

$$\begin{aligned}
\mathcal{I} &= \int dq_1 \left\{ \int_{q_i}^{q_1} \mathcal{D}(q)\, \exp\left(\frac{i}{\hbar}\mathcal{S}(q)\right)\right\} \left\{ \int_{q_1}^{q_f} \mathcal{D}(q)\, \exp\left(\frac{i}{\hbar}\mathcal{S}(q)\right)\right\} q_1 \\
&= \int dq_1 \;\, \langle q_f| \mathcal{U}(t_f - t_1)\, |q_1\rangle\, q_1\, \langle q_1| \mathcal{U}(t_1 - t_i)\, |q_i\rangle \\
&= \langle q_f| \mathcal{U}(t_f - t_1) \mathcal{Q}\, \mathcal{U}(t_1 - t_i)\, |q_i\rangle \\
&= \langle q_f, t_f|\, \mathcal{Q}_H(t_1)\, |q_i, t_i\rangle
\end{aligned}$$

where we have used the diagonalisation of the operator Q in the position basis i.e. $Q = \int dq_1\, |q_1\rangle\, q_1\, \langle q_1|$ and it is understood that we are working in the Schrodinger picture until the last equality where we move to the Heisenberg picture.

What have we done?

We started with a classical measurement where we tried to measure the position at some intermediate time (inserting $q(t_1)$ in the clasical path integral). However, it turns out that this is equivalent to inserting the quantum position operator $\mathcal{Q}_H(t_1)$ inside the path integral in the Heisenberg picture. This is almost like a path integral version of the correspondence principle.

Inserting classical object $q(t_1) \longleftrightarrow$ Quantum operator in Heisenberg picture $\mathcal{Q}_H(t_1)$

Now we look at the insertion of two classical objects $q(t_1), q(t_2)$ with $t_i < t_1, t_2 < t_f$(two measurements of position). Playing the same game, one can insert the Heisenberg picture position operators in the path integral. However, there is now a subtlety. In doing the same decomposition as before, one would need to know whether $t_1 < t_2$ or $t_1 > t_2$. Thus, we would get:

$$\int_{q_i}^{q_f} \mathcal{D}(q)\, \exp\left(\frac{i}{\hbar}\mathcal{S}(q)\right) q(t_1)\; q(t_2) = \begin{cases} \langle q_f, t_f|\, \mathcal{Q}_H(t_1)\;\, \mathcal{Q}_H(t_2)\, |q_i, t_i\rangle & t_1 > t_2 \\ \langle q_f, t_f|\, \mathcal{Q}_H(t_2)\;\, \mathcal{Q}_H(t_1)\, |q_i, t_i\rangle & t_2 > t_1 \end{cases}$$

We therefore see that the path integral naturally gives a notion of time ordering of the objects. We define the 'time ordered product':

$$T\left\{\mathcal{A}(t_1), \mathcal{B}(t_2)\right\} = \begin{cases} \mathcal{A}(t_1)\mathcal{B}(t_2) & t_1 > t_2 \\ \mathcal{A}(t_2)\mathcal{B}(t_1) & t_2 > t_1 \end{cases} \tag{7}$$

(As a caveat, if in the Schrodinger picture $[\mathcal{A}, \mathcal{B}] \neq 0$, then the above notion is not well defined for $t_1 = t_2$.) In this notation, we can rewrite the path integral as:

$$\int_{q_i}^{q_f} \mathcal{D}(q)\, \exp\left(\frac{i}{\hbar}\mathcal{S}(q)\right) q(t_1)\; q(t_2) = \langle q_f, t_f|\, T\left\{\mathcal{Q}_H(t_1), \mathcal{Q}_H(t_2)\right\} |q_i, t_i\rangle \tag{8}$$

Note that in the Heisenberg picture, $[\mathcal{Q}_H(t_1), \mathcal{Q}_H(t_2)] \neq 0$.

2.2 Vacuum States

So far we have not discussed the idea of ground states. We are normally also interested in matrix elements of operators in the ground state. We now go back to working in Euclidean time, so recall that:

$$\mathcal{U}_E(\tau) = \exp\left(-\frac{\tau}{\hbar}\mathcal{H}\right)$$

If we only consider a particle in a (time independent) potential, then we will have energy eigenstates that form a basis. Thus, the vacuum state is defined as the energy eigenstate with the lowest energy eigenvalue ($E_0 < E_1 < .... < E_n < ....$). Then, we can diagonalize the operator as:

$$\mathcal{U}_E(\tau) = \sum_n \exp\left(-\frac{\tau}{\hbar}E_n\right) |n\rangle \langle n|$$

As $\tau \to \infty$, the exponential term suppresses the expansion and the highest contribution comes from the lowest energy eigenvalue i.e.

$$\lim_{\tau\to\infty} \mathcal{U}_E(\tau) \approx \exp\left(\frac{\tau}{\hbar}E_0\right) |0\rangle \langle 0|$$

Defining (ensemble) averages

If $\mathcal{O}$ is some observable and $F(\mathcal{O})$ is some function of the observable, then in a thermal state at (inverse) temperature $\beta = \frac{1}{k_B T}$, we define the average:

$$\langle F(\mathcal{O})\rangle_\beta = \frac{\mathrm{tr}\left\{F(\mathcal{O})\; \mathcal{U}_E(\beta\hbar)\right\}}{Z} = \frac{\mathrm{tr}\left\{F(\mathcal{O})\; \mathcal{U}_E(\beta\hbar)\right\}}{\mathrm{tr}\left\{\mathcal{U}_E(\beta\hbar)\right\}} \tag{9}$$

What would this projection to the ground state, mean for the path integral? Consider the thermal state average of some function of $\mathcal{Q}$:

$$\langle F(\mathcal{Q})\rangle_\beta = \frac{\mathrm{tr}\left\{F(\mathcal{Q})\; \mathcal{U}_E(\beta\hbar)\right\}}{\mathrm{tr}\left(\mathcal{U}_E(\beta\hbar)\right)} = \frac{\int_{\mathrm{PT}} \mathcal{D}[q]\, \exp\left(\frac{-\mathcal{S}_E(q)}{\hbar}\right) F(q(\sigma = 0))}{\int_{\mathrm{PT}} \mathcal{D}[q]\, \exp\left(\frac{-\mathcal{S}_E(q)}{\hbar}\right)} \tag{10}$$

where the PT stands for the periodic boundary conditions for trajectories in Euclidean time. Now, if we want inner products on the vacuum, then one can take the limit where the period is infinite, i.e. at infinite Euclidean time:

$$\langle 0|\, F(\mathcal{Q})\, |0\rangle = \lim_{\tau_\beta\to\infty} \langle F(\mathcal{O})\rangle_\beta$$

2.3 Quantizing the Scalar Field

The scalar field $\phi(x)$ is used to describe relativistic scalar particles (Klein Gordon particles) of mass m with spin $s = 0$ which is non-interacting when we consider the free scalar field, and the particles obey Bose-Einstein statistics i.e. they are bosons. Here $x = (t, \mathbf{x})$ and we work in d dimensions (1 temporal, $d-1$ spatial) with the signature $(-++...+)$.

The classical action is given by:

$$\mathcal{S}_C[\phi] = \int d^d x \left(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right)$$

This is the theory in real time. From the equivalence discussed above, we look at the same theory in Euclidean time $\tau = -it$. Then, the Euclidean action is:

$$\mathcal{S}_E[\phi] = -i\mathcal{S}_C[\phi] = \int d^d x_E \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 \right)$$

where $x_E = (\tau, \mathbf{x})$.

To construct path integrals for this field theory (in real time), we can apply a similar recipe as before and we will obtain a functional integral:

$$\int \mathcal{D}[\phi(x)] \, \exp\left(\frac{i}{\hbar}\mathcal{S}[\phi] \right) = \lim_{\Delta t, \Delta x \to 0} \int \prod_{\tau \in \mathcal{Z}^d} \left[d\phi_\tau \left(\frac{2\pi\hbar\Delta t}{(\Delta x)^{d-1}} \right) \right]^{-1/2} \exp\left(\frac{1}{\hbar}\mathcal{S}_C^{\mathrm{d}}[\phi] \right)$$

where the product suggests that we first look at a path integral for a discrete set of points with $\mathcal{S}_C^{\mathrm{d}}[\phi]$ representing the discretized classical action, and the continuum limits outside help us construct the path integrals for the fields which, to reiterate, are functionals over the field configurations. It is important to note that the action is still quadratic in the fields, which means that the integrals are actually Gaussians (now in the field), so we can carry out the explicit computations using Gaussian integrals.

If we want to carry the above idea to fields at finite temperature, we can work in Euclidean time. Then, we could in principle write down the path integral

$$\int \mathcal{D}[\phi(x_E)] \, \exp\left(-\frac{1}{\hbar}\mathcal{S}_E[\phi] \right)$$

However, there are some subtle issues that one must keep in mind.

1. As mentioned earlier, we now have $x_E = (\tau, \mathbf{x})$. However, since trajectories in Euclidean time are periodic, this means that our background is actually periodic in time (τ) but not in space ($\mathbf{x}$). Thus, the geometry is something like a Circle $\times$ Shape i.e. like a cylinder.

2. The physics here is that at finite temperature we are describing equilibrium thermal states. But **thermal states are not Lorentz invariant.** That is, for a thermal state at some temperature which is in equilibrium with a thermal bath, after a Lorentz transformation, the state may no longer be in equilibrium thus breaking Lorentz invariance.

3. This discrepancy can also be seen by the geometry where firstly Lorentz transformations (rotations in Minkowski) become simple rotations in Euclidean space. However, on a cylinder, we do not have rotation along the unbounded directions. Rotations in the cross section will then correspond to the rotations/periodicity in Euclidean time.

3 Free Scalar Propagator and Wick's Theorem

Recall the theory of the free scalar field, which we claim is obtained from the functional integral given above. We now want to check this claim. To move from QM to QFT, we first note how we can extend the idea of position eigenstates $\{|q\rangle\}$ to QFT. This is done schematically as shown below:

$$\begin{array}{ccc} |q\rangle & \longrightarrow & |\phi\rangle \\ \text{QM} & & \text{QFT} \end{array}$$

where $\phi = \{\phi(\mathbf{x})\}$ is the set of all possible field configurations **on a given time slice**. This is not so easily defined/obtained. Instead we can look at matrix elements of operators for the vacuum or for 'particle states'. For the vacuum state, we saw how this can be done. In Euclidean time, in the limit where the 'period' $\longrightarrow \infty$ we obtain matrix elements in the vacuum state. Motivated by the correspondence principle seen earlier in the context of QM, we expect the following correspondence as well:

$$\text{Inserting classical object } \phi(x) \overset{?}{\rightarrow} \phi(x) \text{ Field operator in the Heisenberg picture}$$

Thus, we use the Euclidean path integral to construct the field operators (in Euclidean time) and then perform a Wick rotation to go back to the theory in real time. We insert $\phi(x_1), \phi(x_2)...\phi(x_n))$ in the Euclidean path integral. We interpret the insertion of these operators as *creating the field out of the vacuum* and then we will check if these fields are the same as those obtained from canonical quantization. Consider,

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{Z}\int \mathcal{D}[\phi]\,\exp\left(-\frac{1}{\hbar}\mathcal{S}_E[\phi]\right)\phi(x_1)\phi(x_2)$$

We define this to be the expectation value of the product of 2 fields at 2 points i.e. the two-point function. This is in complete analogy with the QM case. Computing this, is particularly easy since the $\mathcal{S}_E$ is quadratic in the fields ϕ and this will simply give us Gaussian integrals. With some foresight, we can rewrite the Euclidean action in the following way using integration by parts:

$$\mathcal{S}_E = \frac{1}{2}\int d^d x\ \phi(x)\left(-\Delta_x + m^2\right)\phi(x) + \text{ Unimporant boundary terms}$$

where $\Delta_x = \partial_\mu\partial^\mu$ is the Laplace operator/Laplacian (it is understood that x here is actually x_E).

Gaussian Integrals in Infinite Dimensional Spaces

Actually, one can show that

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{\int \mathcal{D}[\phi]\, \exp\left(-\frac{1}{\hbar}\mathcal{S}_E[\phi]\right)\phi(x_1)\phi(x_2)}{\int \mathcal{D}[\phi]\, \exp\left(-\frac{1}{\hbar}\mathcal{S}_E[\phi]\right)} \text{ is similar to}$$

$$\frac{\int \prod_i d\phi_i\, \exp\left(-\frac{1}{2}\phi_i A_{ij}\phi_j\right)\phi_{i_1}\phi_{i_2}}{\int \prod_i d\phi_i\, \exp\left(-\frac{1}{2}\phi_i A_{ij}\phi_j\right)}$$

where $\{\phi_i\}$ is an N dim vector and A_{ij} is an $N \times N$ symmetric, positive definite matrix. Then,

$$\langle\phi_{i1}\phi_{i2}\rangle = \left(A^{-1}\right)_{i_1 i_2}$$

If we use the above idea, then

$$\langle\phi(x_1)\phi(x_2)\rangle = \langle x_2|\, \frac{1}{-\Delta_x + m^2}\, |x_1\rangle \equiv G(x_2, x_1)$$

But what does the 'inverse' of this Laplacian mean? It means that the function $G(x_2, x_1)$ satisfies the differential equation:

$$\left(-\Delta_x + m^2\right) G(x_2, x_1) = \delta(x_2 - x_1)$$

i.e. $G(x_2, x_1)$ is t he Green's function of the operator $-\Delta_x + m^2$. Now, to understand the solution, first note that the operator $-\Delta_x + m^2$ is symmetric under translations and rotations. Therefore, $G(x_2, x_1) = G(x_2 - x_1) = G(|x_2 - x_1|)$. We can better understand the solution in Fourier space, so we define the Fourier transform

$$\hat{G}(K) = \int d^d x\, \exp\left(-iK \cdot X\right) G(X)$$

To understand the boundary conditions of the differential equation, we can first consider a spacetime that is periodic in time with period T and is also periodic in space with period L (along each axis). That is, we consider a toroid $T \times L^{d-1}$. With this background, the Euclidean action will be exactly given by the first term with the Laplace operator and will not have boundary terms. Then, one can perform the integration as discussed above. Once this is computed, we can take the limit where $T, L \to \infty$. Then, the boundary conditions for $G(X)$ are such that the solutions must decay at infinity. To look at $\hat{G}(K)$, we can Fourier transform the differential equation for $G(X)$ and that gives us:

$$\left(K^2 + m^2\right)\hat{G}(K) = 1 \implies \hat{G}(K) = \frac{1}{K^2 + m^2}$$

Thus, we have the solution:

$$G(X) = \int \frac{d^d K}{(2\pi)^d}\, \exp\left(iK \cdot X\right) \frac{1}{K^2 + m^2}$$

In fact, another way to compute this is to simply extend the symmetry arguments that were used to restrict the form of G to $G(|x_2 - x_1|)$. Fixing $r = |x_2 - x_1|$, we can expand the Laplacian and we see that only the radial part is unknown, which gives us the equation:

$$\left[-\left(\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}\right) + m^2\right] G(r) = \delta(r)$$

where the 'origin' $r = 0$ is still not well defined. However, this is a well known ODE with Bessel functions as the solutions. With the BC suggesting that the Green's function decays at infinity, we have the solution:

$$G(r) = \frac{1}{2\pi}\left(2\pi\frac{r}{m}\right)^{\frac{2-d}{2}} K_{\frac{d-2}{2}}(rm)$$

where $K_y(z)$ is the Bessel function of the 2nd kind. As special cases:

$$\lim_{m\to 0} G(r) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}}\, r^{2-d}$$

$$G(r) \to \begin{cases} \exp(-rm) & \text{At long distances with } m > 0 \\ \begin{cases} r^{2-d} & d > 2 \\ -log(r) & d = 2 \end{cases} & \text{At small r (setting } m = 0) \end{cases}$$

Note that we set $m = 0$ in the small r limit since in this regime, the mass term is simply a correction in the ODE and the equation is dominated by the $\frac{1}{r}$ term. We can see that the solution is singular in the short distance limit. We reiterate that we are still working in Euclidean spacetime. If we move to Minkowski spacetime, the Laplace operator in Euclidean is replaced by that in Minkowski, which gives us the equation:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla + m^2\right) G(X) = -i\delta(X)$$

The $-i$ comes from the fact that the temporal delta function in Euclidean space is $\delta(\tau) = -i\delta(t)$. We recognise the above differential equation as the equation of the propagator in QFT. In particular, this actually gives us the Feynman propagator. To see this, we go back to the solution in Euclidean time. Here we have:

$$G_E(X) = \int \frac{dk_0}{2\pi}\frac{d\mathbf{k}}{(2\pi)^{d-1}}\, \exp\left(ik_0\tau + i\mathbf{k}\cdot\mathbf{x}\right)\frac{1}{k_0^2 + \mathbf{k}^2 + m^2}$$

When we Wick rotate to go to Minkowski spacetime, we take $\tau = it$. But to ensure that the exponential in the Fourier transform remains well defined, we will also have to (counter) Wick rotate the k_0 variable, which is allowed since it is anyway a dummy variable. Suppose we do this as $k_0 = \frac{1}{i}\omega$. Then, the above solution becomes:

$$G(X) = \int \frac{d\omega}{2\pi}\frac{d\mathbf{k}}{(2\pi)^{d-1}}\, \exp\left(i\omega t + i\mathbf{k}\cdot\mathbf{x}\right)\frac{-i}{-\omega^2 + \mathbf{k}^2 + m^2}$$

Even in Euclidean spacetime, the propagator has poles at $k_0 = \pm i(\mathbf{k}^2 + m^2)^{1/2}$. Therefore, in Euclidean spacetime, the poles are along the Y axis so the contour must be deformed to avoid the poles on the Y Axis. Upon Wick rotation, these deformations are detours around the X axis, in exactly the same way as it is for the Feynman propagator. This is the idea behind claiming that the propagator obtained in Minkowski space is the Feynman propagator.

(Insert the diagram)

Why is this natural?

Recall that in real time, inserting two operators in the path integrals gives us a time ordered product. Therefore, when we compute the two point function, we obtain the time ordered product of the field operators (which are what we insert to compute the two-point function). Therefore,

$$\begin{aligned} \langle \phi(x_1)\phi(x_2)\rangle &\longrightarrow \langle \Omega| \, T\{\phi(x_1), \phi(x_2)\} \, |\Omega\rangle \\ \text{Minkowski Signature } &\longrightarrow \text{After Wick rotation} \end{aligned}$$

It is important to remember that

$$[\phi(x_1), \phi(x_2)] = \begin{cases} \text{Non Zero} & \text{if } x_1, x_2 \text{ are timelike separated} \\ 0 & \text{if } x_1, x_2 \text{ are spacelike separated} \end{cases}$$

However, if $x_1 = x_2 \implies x_1^0 = x_2^0 = t_1 = t_2$. In the solution for the propagator discussed above, we saw that at $r = 0$ the solution diverges. This is actually related to the fact that the commutation relations at different times go to zero. However, if we are in only one dimension, then the Green's function also does not diverge. We can still do QFT in 1D which means there is only time, no space so it's like a quantum mechanical system.

3.1 Wick's Theorem

To see further, the relationship between two-point functions from the path integral and the Feynman propagator, we can look at higher n point correlation functions. So far, we are working with fields which have an action that resembles the Gaussian distribution. Then, we use the following general feature of (functional) Gaussian integrals:

$$\begin{aligned} \langle \phi_{i_1}\phi_{i_2}\phi_{i_3}\phi_{i_4}\rangle &= \langle \phi_{i_1}\phi_{i_2}\rangle\langle \phi_{i_3}\phi_{i_4}\rangle + \langle \phi_{i_1}\phi_{i_3}\rangle\langle \phi_{i_2}\phi_{i_4}\rangle + \langle \phi_{i_1}\phi_{i_4}\rangle\langle \phi_{i_2}\phi_{i_3}\rangle \\ \text{That is } \langle \phi_{i_1}\phi_{i_2}\phi_{i_3}\phi_{i_4}\rangle &= \sum_{\text{all pairs}} \langle \phi\phi\rangle\langle \phi\phi\rangle \end{aligned}$$

We can see how this result relates to four-point functions and Green's functions in QFT. That is, we have:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle\phi(x_1)\phi(x_2)\rangle\langle\phi(x_3)\phi(x_4)\rangle + \langle\phi(x_1)\phi(x_3)\rangle\langle\phi(x_2)\phi(x_4)\rangle + \langle\phi(x_1)\phi(x_4)\rangle + \langle\phi(x_2)\phi(x_3)\rangle$$

However, this is simply Wick's theorem (which holds whether we are in Minkowski or Euclidean time). Thus, we have obtained Wick's theorem as a general feature of the path integral formalism.

References

[1] J. Sakurai and J. Napolitano, "Modern quantum mechanics," 2017.