Superradiance in Black Hole Spacetimes

An M.Sc thesis report in partial fulfilment of the M.Sc. Physics program

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DECLARATION

I hereby declare that the work presented in this report titled **Superradiance in black hole spacetimes** submitted by me for the evaluation of PH509/PH518 course-work in in the Master of Science, Physics program at IIT Guwahati, is an authentic record of the work carried out under the guidance of Prof Sayan Chakrabarti. Due references and acknowledgments have been given in the report to all the materials used.

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CERTIFICATE

This is to certify that the above information given by the candidate is correct to the best of my knowledge.

Prof Sayan Chakrabarti **Date: 24/04/22**

Abstract

Black holes are classical solutions to Einstein's field equations arising in general relativity. Superradiance is a known phenomenon in various areas of physics where one observes radiation amplification, in different contexts. In the context of black hole superradiance there have been efforts to understand the scattering of different fields, both classically and quantum mechanically, against black hole horizons and study their reflection and transmission coefficients in an effort to understand any possible amplification. In this report, we begin by reviewing the basic ideas of superradiance in two classical - the Reissner-Nordström and the Kerr black hole. We study the conditions of superradiant amplification in both these settings and look at the explicit amplification factorss. Amplification factors for scalar fields are also numerically obtained. We extend the application of this formalism to the newly proposed loop quantum gravity inspired Ashtekar Olmedo Singh black holes by considering rotating solutions constructed using the modified Newman Janis algorithm. We discuss the horizon structure, scalar field dynamics in this background and superradiant scattering of scalar field. We compute the amplification factors analytically and discuss their behaviour.

Keywords: superradiance, black holes, spacetime, radiation, differential equations, dissipation, general relativity, quantum mechanics

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1 Introduction

Superradiance is a radiation amplification phenomenon which is observed in many different contexts and in many areas in physics. In high energy physics, Klein's paradox [1] was an early example of superradiance when relativistic particle mechanics tried to explain how scattering processes sometimes resulted in a reflected number density that was greater than the incident density. A common feature in examples of superradiance is the presence of some dissipative mechanism. In the case of black holes, the event horizon represents a dissipative surface in the sense that there is a fundamental irreversibility at the event horizon which acts as a one way membrane.

A large source of reference for this report and contemporary work in black hole superradiance is this recent review [2]. An earlier review of superradiance especially in the context of general relativity and black holes was given by Bekenstein in [3]. Some of the earlier developments were the introduction of a new formalism to understand scattering phenomena in black hole spacetimes. For instance, Newman and Penrose introduced a special tetrad formalism of general relativity using spin coefficients [4]. This was used by Teukolsky in order to separate the field equations [5] which is something that will be discussed in Section 4. A curious feature is the fact that fermions (both massless [6], [7] and massive [8], [9], [10]) do not show superradiance.

It was also shown by Zeldovich that incident waves on any rotating body with some kind of dissipation will be amplified [11], [12]. Zeldovich was also able to show that quantum mechanically rotating bodies will also cause pair production. These results were used by Hawking to propose the idea of BH evaporation [13] which is one of the earliest ideas of quantum field theories in curved space time. Subsequently, these ideas led to the notion of black hole entropy [14], and work began on understanding black hole thermodynamics.

An early example of black hole superradiance is the Penrose process [15], wherein Penrose showed that the ergoregion of Kerr black holes can aid in a special scattering process through which energy can be extracted out of the black hole. He showed that if a particle entered the ergoregion from outside and disintegrates into two particles, then it can be arranged such that one of the decay products has negative energy and is hence engulfed by the black hole. However, energy conservation then dictates that the other decay product must have a positive energy greater than that of the incident particle. This is a characteristic feature of superradiance. In this context, we will look at scattering of some arbitrary field against black holes, and determine the conditions under which the fields undergo superradiant scattering. In this superradiant regime, we can also compute the amplification factors for the fields.

While the Penrose process looks at particle scattering against black holes, one can easily consider fields in a similar setting. Various spin fields scattering against black holes spacetimes can also give rise to superradiance (although as mentioned before, Dirac fields do not exhibit superradiance in the classical Kerr geometry). Throughout this thesis, we work with fields, and scalar fields in specific. This report is arranged as follows. In Section 2 we start with a survey of superradiance around black hole solutions, and how one determines the conditions for superradiance. In Sec 3, we consider the Reissner-Nordström geometry and look at superradiance in this static geometry. In Section 4 we discuss in detail superradiance in Kerr. We look at analytic and numerical methods to solve for the amplification factors, which will also be used in later sections. In Section 5, we consider the recently proposed Ashetkar Olmedo Singh black holes inspired by loop quantum gravity. Using rotating solutions in this theory constructed through the modified Newman Janis algorithm [16], we compare superradiance between the classical and the quantum corrected rotating black hole backgrounds.

2 Overview of Black Hole Superradiance

Suppose we have some field Ψ that is incident on a black hole described by a metric $g_{\mu\nu}$. We can then write down an action that can be written in the form:

$$\mathcal{S} = \frac{1}{\kappa}\int \sqrt{-g}\; d^4x R + \mathcal{S}_M \equiv \mathcal{S}_{\text{grav}} + \mathcal{S}_M \tag{1}$$

where

1. g is the determinant of the metric
2. $\mathcal{S}_M$ is the matter action that contains the field Ψ
3. R is the Ricci scalar built from contractions of the Riemann Curvature tensor
4. $\kappa = 16\pi$

Until Section 5 we consider units where $G = c = 1$. Throughout this report we use the metric convention $(-+++...)$ and take the cosmological constant $\Lambda = 0$.
Varying the action with respect to the fields Ψ is equivalent to varying the matter action $\mathcal{S}_M$ and this will gives us equations of motions for the field. Thus, the dynamics of the field are governed by these equations.

If we consider a massless charged scalar field Ψ in some curved spacetime as an example then we can write (1) as:

$$\mathcal{S} = \int d^4x\; \sqrt{-g}\left(\frac{R}{\kappa} + g^{\mu\nu} D_\mu \Psi^* D_\nu \Psi\right)$$

where $F_{\mu\nu}$ is the Maxwell field tensor for the electromagnetic fields due to the charged field and $D_\mu = \nabla_\mu - iqA_\mu$ is the minimally coupled covariant derivative.
Varying the action, or using the Euler Lagrange equations for the field Ψ we get:

$$(\nabla_\mu - iqA_\mu)(\nabla^\mu - iqA^\mu)\,\Psi = 0 \tag{2}$$

In order to solve this we assume a stationary axisymmetric background which allows us to consider an ansatz for the field as:

$$\Psi(t, r, \theta, \phi) = \int d\omega \sum_{lm} e^{-i(\omega t)} Y_{lm}(\theta, \phi)\frac{\psi(r)}{r}$$

Using this ansatz, we can separate the equations for the function $\psi(r)$ and bring it to a form given by:

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}r_*^2} + V_{\text{eff}}\;\psi(r) = 0 \tag{3}$$

This is a Schrödinger like equation in the coordinate r_* which is called the tortoise coordinate. The function $\psi(r)$ can in principle be complex. The potential V_{eff} depends on the field and the metric of the surrounding spacetime. We take the potential to become If the wave number k near the horizon and at infinity is denoted by k_H and k_∞ then, the function ψ has the following asymptotic form

$$\psi = \begin{cases} Te^{-ik_H r_*} & \text{for } r \to r_H \\ Ie^{-ik_\infty r_*} + Re^{ik_\infty r_*} & \text{for } r \to \infty \end{cases} \tag{4}$$

If we consider (3) and take the complex conjugate, we can get a second equation for $\overline{\psi}$. Treating ψ and $\overline{\psi}$ as linearly independent functions and taking the asymptotic form of $\overline{\psi}$ from (4), we can evaluate the Wronskian for the two functions given by:

$$\mathcal{W} = \begin{vmatrix} \psi(r) & \overline{\psi(r)} \\ \psi'(r) & \overline{\psi'(r)} \end{vmatrix}$$

We can compute this Wronskian at both the extreme locations. As $r \to r_H$, we have

$$\begin{aligned} \psi &= Te^{-ik_H r_*}, \\ \overline{\psi} &= T^* e^{ik_H r_*}, \\ \psi' &= -ik_H T e^{-ik_H r_*}, \\ \overline{\psi} &= ik_H T^* e^{ik_H r_*}. \end{aligned}$$

Hence,

$$\mathcal{W}_H = 2ik_H |T|^2 \tag{5}$$

Similarly, the Wronskian at infinity can also be found which is given by:

$$\mathcal{W}_\infty = 2ik_\infty \left(|I|^2 - |R|^2\right) \tag{6}$$

Since the Wronskian must be independent of the value of r_* at which it was calculated, we can equate (5) and (6) and hence we have the relation:

$$|R|^2 = |I|^2 - \frac{k_H}{k_\infty}|T|^2 \tag{7}$$

The relation between the amplitude of the coefficients in a scattering wavefunction generally give us information about the reflected and transmission coefficients. Analogous to that, we see that:

$$\text{If } \frac{k_H}{k_\infty} < 0 \implies |R|^2 > |I|^2 \tag{8}$$

Thus, we have a situation where the reflected amplitude is greater than the incident amplitude. This is the condition and the definition of superradiance. Depending on the background considered, the potential V_{eff} will change, thus changing the form of k_H, k_∞. Thus, by using (8), one can find the superradiance condition.

Now that we have a procedure to determine the possibility of superradiance, we consider specific background geometries, starting with the Reissner-Nordström geometry in the following section.

3 The Reissner-Nordström Geometry

The Reissner-Nordström solution is one of the spherically symmetric solutions of the Einstein field equations. This metric represents the spacetime *outside* a static, spherically symmetric charged black hole. We will consider the propagation of a scalar field in this spacetime, and look at the superradiance features applicable in this case. The action for this background is given by:

$$\mathcal{S} = \int \sqrt{-g}\; d^4x \left(\frac{R}{\kappa} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

3.1 The Reissner-Nordström spacetime

The line element for an RN black hole of M and charge Q is given by:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 \tag{9}$$

with

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{10}$$

Note that (9) is valid generally for any static spherically symmetric metric. Hence, the function $f(r)$ is sufficient to describe a spherically symmetric, static spacetime. The tortoise coordinate which was mentioned earlier is defined as:

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{1}{f(r)}$$

This is generally not an invertible transformation. Given the metric in (9),(10) we want to calculate the potential V_{eff} and thus, the Schrödinger like equation so as to calculate the superradiance condition.

3.2 Scattering in Reissner-Nordström spacetime

If we consider a massless scalar field incident on the RN black hole, then the dynamics of the scalar field is described by (2) - the Klein Gordon equation. We use the ansatz mentioned above, for a wave of mode ω:

$$\Psi(t, r, \theta, \phi) = \sum_{lm} e^{-i(\omega t)} Y_{lm}(\theta, \phi) \frac{\psi(r)}{r} \tag{11}$$

The Klein Gordon equation can be written as $\Box\Psi = 0$ which can be expanded using the identity:

$$\Box = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \right) \tag{12}$$

Using this, the effective potential is given by:

$$V_{\text{eff}} = \omega^2 - f\left(\frac{l(l+1)}{r^2} + \frac{f'}{r} \right) - \frac{2qQ\omega}{r} + \frac{q^2 Q^2}{r^2} \tag{13}$$

If we now look at asymptotic solutions to (3) then we can have plane wave solutions given by $e^{\pm ikr_*}$ as in (4), where the wave vector k is defined in terms of the effective potential as:

$$k \sim \sqrt{V_{\text{eff}}}$$

The appropriate limits on k can then be taken, by applying the limits to V_{eff}. Thus, using (13) and

$$f(r) \to \begin{cases} 0 & \text{for } r \to r_H \\ 1 & \text{for } r \to \infty \end{cases} \tag{14}$$

Then we have:

$$k_H = \sqrt{V_H} = \sqrt{\omega^2 - \frac{2qQ\omega}{r_+} + \frac{q^2Q^2}{r_+^2}} = \omega - \frac{qQ}{r_+}$$

Similarly,

$$k_\infty = \sqrt{V_\infty} = \omega$$

If we use the condition given in (8), we have:

$$\frac{k_H}{k_\infty} = \frac{\omega - qQ/r_+}{\omega} \tag{15}$$

Hence, the superradiance condition is given by $k_H < 0$ which means that:

$$\omega < \frac{qQ}{r_+}$$

The same condition can also be derived by thermodynamic arguments as given in [17] and [2].

3.3 Amplification Factors

We define the amplification factor for the scalar field as:

$$Z_{\text{slm}} = \frac{|R|^2}{|I|^2} - 1$$

where s is the spin of the incident field, l orbital quantum number of the field, m azimuthal quantum number of the field. In order to compute the amplification factors explicitly, we will have to solve the Schrödinger like equation given in (3) with the potential from (13). This is done numerically in the code written by the authors of [17] which can be found here. In this program, the algorithm followed is a matching procedure where the Schrödinger like equation is Taylor expanded and approximately solved near the horizon and at infinity, corresponding to the asymptotic solutions given in (4). Then, this is matched, term by term, to the asymptotic solution and to required order, the coefficients R, I, T are determined. This can be used to solve for the amplification factors. A detailed description of the matching procedure is also included in the rotating case that will be considered in Section 4, where this procedure is used in analytical calculation of the amplification factors. The dependence of Z_{000} on ω which is obtained numerically using the above mentioned program is given in Figure 1.

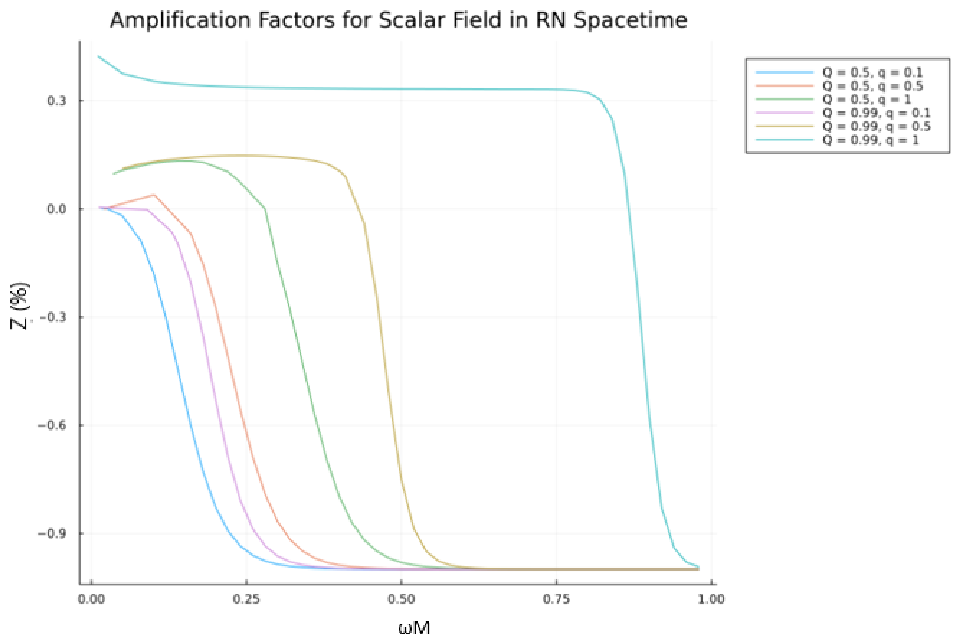

Figure 1: Superradiant amplification factor for $l = 0$ waves of scalar field in RN spacetime. Consistent with the condition in (15), we see that with an increase in Q there is a greater range of ω for which $Z_{000} > 0$ which suggests superradiant amplification

3.4 Numerical Approach to Amplification Factors

The plot above is obtained using the program written by the authors of [2] which can be found here. The numerical approach followed here is quite similar to the one followed for the rotating case as well, which is mentioned in Section 4.5. The aim is to solve the Klein Gordon equation in the Schrödinger like form, approximately, by making a Taylor expansion at the horizon and at infinity. Then, the same equation is solved numerically using Mathematica and these two solutions are matched, in order to obtain the scattering coefficients R, I and T.

The effective potential considered for the RN case, is the one given in (13). The tortoise coordinate which is usually defined as

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{1}{f(r)}$$

In order to write the coordinate transformation explicitly, one has to integrate the radial function which is a part of the metric. In general, this cannot be done exactly and approximately (near horizon) is given by:

$$r_* = r + M \log{(r_+)(r_-)} + \frac{2M^2 - Q^2}{2\sqrt{M^2 - Q^2}} \log \frac{r - r_+}{r_-}$$

This can be done by integrating the transformation while Taylor expanding the function $f(r)$ at the horizon. The ODE given in the code is:

$$f^2(r)\psi''(r) + f'(r)f(r)\psi'(r) + V_{\text{eff}}\psi(r) = 0$$

This can be obtained from the Klein Gordon equation as follows.

The radial part of the Klein Gordon equation can be written as

$$\Delta\frac{d}{dr}(\Delta\frac{dR}{dr}) + U(R) = 0$$

where $\Delta = r^2 - 2Mr + Q^2 = r^2 f$ and $U = r^4[(\omega - \frac{qQ}{r})^2 - f(l(l+1))]$.

Expanding this out, we get:

$$f^2R'' + ff'R' + \left(\omega - \frac{qQ}{r}\right)^2 R - f\left(l(l+1) + \frac{f'}{r}\right)R + \left[\frac{2f^2}{r}R' + \frac{ff'}{r}R\right] = 0$$

That is,

$$f^2R'' + ff'R' + V_{\text{eff}}(r)R + \left[\frac{2f^2}{r}R' + \frac{ff'}{r}R\right] = 0$$

Now we substitute $\psi(r) = rR(r)$. Therefore,

$$f^2\left(\frac{\psi''}{r} - \frac{2\psi'}{r^2} + \frac{2\psi}{r^3}\right) + ff'\left(\frac{\psi'}{r} - \frac{\psi}{r^2}\right) + V_{\text{eff}}(r)\frac{\psi}{r} + \left[\frac{2f^2}{r}(\frac{\psi'}{r} - \frac{\psi}{r^2}) + \frac{ff'}{r}\frac{\psi}{r}\right] = 0$$

Simplifying we have the final ODE:

$$f^2\psi'' + ff'\psi' + V_{\text{eff}}(r)\psi = 0 \tag{16}$$

where throughout this analysis we assume that r is related to r_* as described above, since the equation is to be solved in the tortoise coordinate. Then, at the horizon we consider an ansatz of the form:

$$\psi = e^{ikr_*}h(r)$$

Once this is substituted in the ODE, we can eliminate all the dependence on r_* and solve for the function $h(r)$. This is done by expanding $h(r)$ at the horizon up to desired order as:

$$h(r) = \sum_n a_n(r - r_+)^n$$

Substituting in the ODE and collecting terms for each power of $(r - r_+)$, we get a set of equations (one for each power of r whose coefficients are the unknowns)
Similarly, at infinity, one can expand h as follows:

$$h(r) = \sum_n a_n r^{-n}$$

Thus, we will have two sets of equations, one each for the horizon and at inifinity for which we have coefficients for each power of r in the solution. The ODE given in (16) is then solved numerically using NDSolve on Mathematica and the solution is again expanded at the horizon and infinity in order to match the coefficients with those obtained by the approximate analytic solution. From these, one can determine R, T, I and hence the amplification coefficients.

So far, we have been able to make use of the spherical symmetry present in the background to separate equations, and look at perturbations to the black hole. In the following section, we will look at black hole solutions which are axisymmetric, and this change makes the mathematics considerably harder. We will discuss a different formalism to understand perturbations in the classical rotating (Kerr) background.

4 Kerr Geometry

In this section we look at the process of superradiance and the scattering of different spin fields on a rotating black hole described by the Kerr spacetime. The Kerr geometry describes spacetime *outside* a rotating, uncharged, axisymmetric black hole.

4.1 The Kerr Spacetime

Fundamentally the line element may be written (in Boyer Lindquist coordinates) as follows:

$$ds^2 = \frac{-\Delta}{\rho}\left(dt - a\sin\theta^2 d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin\theta^2}{\rho^2}\left(adt - (a^2+r^2)d\phi\right)^2 \tag{17}$$

where

$$\Delta = r^2 + a^2 - 2Mr$$
$$\rho^2 = r^2 + a^2\cos\theta^2$$

This line element represents the spacetime around an axially symmetric rotating black hole of mass M and angular momentum given by $J = aM$. We have taken $\Lambda = 0$ and thus, the event horizon of the black hole is found as the roots of $g^{rr} = 0$ which in this case would correspond to $\Delta = 0$. We define them to be r_+ and r_- for the two horizons. Further, in the rotating case (unlike in the Schwarzchild metric) $g^{rr} = 0$ and $g_{tt} = 0$ do not give the same surfaces. Instead, the roots of $g_{tt} = 0$ give the surfaces known as the ergoregion, which is different from the event horizons described at the coordinates $r = r_\pm$ as above. We can see that as $a \to 0$ the line element given by (17) reduces to the Schwarzhild solution, thus showing that with no rotation we recover spherical symmetry. This is also an asymptotically flat metric.

4.2 Scattering in Kerr spacetime

In order to consider scattering of fields in this spacetime, we first look at an alternative formulation of general relativity which makes the discussion mathematically simpler. In the line elements and tensors written so far, we have been using coordinate bases. Alternatively, we could have worked with non-coordinate bases and done all of the same physics. At each spacetime point, we could set up a coordinate system, thus choosing four vectors - a tetrad. Based on the symmetrise of the problem, one could choose any kind of tetrad and here, we choose a specific complex null tetrad. This formulation is called the Newman Penrose formalism [4], and this simplifies the Einstein equations in Kerr spacetime. A very brief overview is given below and more details can be found in [18].

4.2.1 Newman Penrose Formalism

Suppose we have a vector v^μ in coordinate basis. We want to project this object on to the coordinate system erected in order to write it in terms of the new non-coordinate basis. This can be done using a transformation

$$v^{(a)} = e_\mu^{\ (a)} v^\mu$$

with $a = 0, 1, 2, 3$ and this index is in parenthesis to distinguish the tetrad index a from the coordinate index μ. The metric itself in this tetrad basis can then be defined as

$$\eta^{(a)(b)} = e_{\mu}^{\ (a)} e_{\nu}^{\ (b)} g^{\mu\nu}$$

Choosing the transformation $e_{\mu}^{(a)}$ carefully can ensure that $\eta^{(a)(b)}$ is the Minkowski metric, or some other suitably easier metric to work with. In this case, we choose four vectors where l^{μ}, n^{μ} are two real null vectors and $m^{\mu}, \overline{m^{\mu}}$ is a complex conjugate null vector pair such that

$$\begin{aligned} &l_{\mu} l^{\mu} = n_{\mu} n^{\mu} = m_{\mu} m^{\mu} = l_{\mu} m^{\mu} = n_{\mu} m^{\mu} = 0 \\ &l_{\mu} n^{\mu} = -m_{\mu} \overline{m^{\mu}} = 1 \end{aligned}$$

This gives us a constant $\eta^{(a)(b)}$, although it will have off diagonal terms and will not be the Minkowski metric. The Ricci rotation coefficients are now called spin coefficients. For a detailed description of the Newman Penrose formalism, one can refer to [18].
The explicit tetrad for the Kerr case is given in [19]. We consider the scattering of fields in Kerr spacetime, in this basis. For an arbitrary field of spin s where

1. $s = 0$ represents a scalar field
2. $s = \pm\frac{1}{2}$ represents mass less Dirac fields
3. $s = \pm 1$ represents an EM field
4. $s = \pm 2$ represents a gravitational field

one can write down the Einstein equations in the tetrad basis. The fields themselves are expressed using Newman Penrose scalars which are projections of the fields into this basis and can be found in [2], [18]. The perturbation for such an arbitrary field can be given by a single master PDE which was derived by Teukolsky, Press and others [5], [19], [20] and is given by

$$\begin{aligned} &\left[\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin\theta^2\right] \frac{\partial^2 \psi}{\partial t^2} + \frac{4Mar}{\Delta} \frac{\partial^2 \psi}{\partial t \partial \phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin\theta^2}\right] \frac{\partial^2 \psi}{\partial \phi^2} \\ &\quad - \Delta^{-s} \frac{\partial}{\partial r}\left(\Delta^{s+1} \frac{\partial \psi}{\partial r}\right) - \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin\theta^2}\right] \frac{\partial \psi}{\partial \phi} \\ &\quad\quad - 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right] \frac{\partial \psi}{\partial t} + (s^2 \cot\theta^2 - s)\psi = 0 \end{aligned}$$

Here, the function $\psi \equiv \psi(t, r, \theta, \phi)$ is related to the Newmann Penrose scalars. In the case of scalar fields, the associated Newman Penrose scalar is the scalar field itself. We take an ansatz for ψ as

$$\psi = \frac{1}{2\pi} e^{-i\omega t} e^{im\phi} S(\theta) R(r)$$

This master equation can be separated into radial and angular equations and was done by Teukolsky, which are given by:

$$\Delta^{-s}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{s+1}\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\left(\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-\lambda\right)R=0 \tag{18}$$

and

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right)+(a^2\omega^2\cos\theta^2-\frac{m^2}{\sin\theta^2}-2a\omega\cos\theta-\frac{2ms\cos\theta}{\sin\theta^2}-s^2\cot\theta^2+s+A_{slm})S=0 \tag{19}$$

where $K\equiv(r^2+a^2)\omega-am$ and $\lambda=A_{slm}+a^2\omega^2-2am\omega$. Here A_{slm} represent the angular eigenvalues and when $a\omega<<1, A_{slm}=l(l+1)-s(s+1)$ The angular equation given by (19) is solved by the set of functions known as the spin-weighted spheroidal harmonics $e^{im\phi}S\equiv S_{slm}(a\omega,\theta,\phi)$ In the limit $a\omega\to 0$ they reduce to the spin-weighted spherical harmonics which are given by $Y_{slm}(\theta,\phi)$

We wish to solve this radial equation, for the slowly rotating case i.e when $a\omega<<1$. In this case, the asymptotic behaviour of the function as $r\to\infty$ is given by:

$$R_{slm}\equiv R(r)\sim I_s\frac{e^{-i\omega r}}{r}+R_s\frac{e^{i\omega r}}{r^{2s+1}} \tag{20}$$

The purpose of the analysis is now to solve the radial equation to try to compute the superradiance amplification factors

4.3 The Radial Equation

The first task is to solve the radial equation above. We have:

$$\Delta^{-s}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{s+1}\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\left(\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-\lambda\right)R=0$$

Since $r_\pm$ are the roots of $g^{rr}=0\implies\Delta=0$, we can write $\Delta=(r-r_+)(r-r_-)$. Then, the radial equation can be written in terms of derivatives of $R(r)$ by simplification and this is given by:

$$\begin{aligned}&\Delta(r)R''(r)+(s+1)R'(r)\Delta'(r)+\\&\left(\frac{(\omega(r^2+a^2)-ma)}{\Delta}\big(\omega(r^2+a^2)-ma)-2is(r-M)\big)+4isr\omega+s(s+1)-l(l+1)\right)R(r)=0.\end{aligned}$$

We can write this equation in the Schrödinger like form in the following way. To do this, first we define the tortoise coordinate r_*:

$$\frac{\mathrm{d}r}{\mathrm{d}r_*}=\frac{\Delta}{r^2+a^2}\equiv\mu(r). \tag{21}$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}r}=\frac{1}{\mu(r)}\frac{\mathrm{d}}{\mathrm{d}r_*}.$$

Using this, and simplifying the radial equation with $\Phi = \sqrt{r^2+a^2}R$, we have:

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}r_*^2} + V_{\text{eff}}\,\Phi = 0, \tag{22}$$

where

1. $V_{\text{eff}}= \left(\frac{K^2-2is(r-M)K+\Delta(4is\omega r-\lambda)}{(r^2+a^2)^2} - u(r)^2 - \mu u'(r)\right)$ which is implicitly a function of r_* through (21)
2. $u(r) = \frac{\mu r}{(r^2+a^2)^2} + \frac{s(r-M)}{(r^2+a^2)}$
3. $K = (r^2+a^2)\omega - ma$

From V_{eff}, one can see that as $r \to \infty$ the only term that is nonzero is $\left(\frac{K}{r^2+a^2}\right)^2 \to \omega^2$. Similarly, as $r \to r_+$, $\left(\frac{K}{r^2+a^2}\right)^2 \to (\omega - m\Omega_H)^2$, with $\Omega_H = \frac{a}{(r_+)^2+a^2}$. Thus, the asymptotic solution to the Schrödinger like form are verified as long as we only consider in going wave modes at the horizon, or no reflection at the horizon. Therefore, we have (in the same notation as in the Schrödinger like equation):

$$\frac{k_H}{k_\infty} = \frac{(\omega - m\Omega_H)}{\omega^2}.$$

This gives us the following condition for sueprradiance amplification:

$$\frac{k_H}{k_\infty} < 0 \implies \omega < m\Omega_H.$$

In solving the radial equation, we only consider the slowly rotating regime and take $a\omega << 1$. In this limit, $\lambda = A_{slm} = l(l+1) - s(s=1)$. Now we define a new coordinate given by

$$x = \frac{r - r_+}{r_+ - r_-}$$

We transform the equation above as an ODE in x, and to do so we see that $\Delta\frac{\mathrm{d}}{\mathrm{d}r} = (r_+ - r_-)x(x+1)\frac{\mathrm{d}}{\mathrm{d}x}$. Using this the above ODE becomes:

$$x^2(x+1)^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + (s+1)x(x+1)(2x+1)\frac{\mathrm{d}R}{\mathrm{d}x} + \left[k^2x^4 + 2iskx^3 - \lambda x(x+1) - isQ(2x+1) + Q^2\right]R = 0 \tag{23}$$

where we have defined the constants

$$\begin{aligned} Q &= \frac{\omega - m\Omega_H}{4\pi T_H}, \\ 4\pi T_H &= \frac{(r_+ - r_-)}{r_+^2}, \\ k &= \omega(r_+ - r_-), \\ T_H &= \frac{a}{r_+^2 + a^2}. \end{aligned}$$

Here, T_H is the temperature of the black hole.
To proceed beyond this point we begin making a few approximations. The general idea would be to follow a 'matching procedure' which is illustrated by the following algorithm

a. Obtain the near horizon solution - a solution for $x << 1$.

b. Obtain the far horizon limit - a solution for $x >> 1$

c. Compare the large x limit of (a) and small x limit of (b) and demand that these solutions must be equal due to the continuity of ψ across space time. This will give us the integration constants from one of the solutions.

d. The far solution can be compared with the asymptotic solution of the Schrödinger like equation from which we can read off the coefficients I, R.

Following the above matching procedure we first try to solve for approximate solutions to the radial equation in the near horizon and far horizon approximation. The full discussion of this solution is given in Appendix A

4.3.1 Near Horizon Limit

In this regime, we take $r - r_+ << 1$. In the newly defined coordinate x this would be equivalent to taking $kx << 1$. Using this approximation in the above equation, we get:

$$x^2(x+1)^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2}+(s+1)x(x+1)(2x+1)\frac{\mathrm{d}R}{\mathrm{d}x}+\left[Q^2-(l(l+1)-s(s+1))x(x+1)-isQ(2x+1)\right]R=0 \tag{24}$$

where λ is written in terms of l, s.

The solution to this equation is

$$R(x) = A_1 x^{-s-iQ}(x+1)^{s-iQ}F\left(-l-s, l-s+1, l-s+1, 1-s-2iQ, -x\right) \tag{25}$$

where $F(a, b, c, x)$ is the hypergeometric function and A_1 is a constant of integration.

4.3.2 Far Horizon Limit

We now solve the radial equation at large distance away from the horizon. If we take $x >> 1$ then we can make the approximation $(x+1) \approx x$. Therefore, the original radial equation can be reduced to:

$$x^4\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + 2(s+1)x^3\frac{\mathrm{d}R}{\mathrm{d}x} + \left[k^2x^4 + 2iskx^3 - \lambda x^2 - 2isQ(x+Q^2\right]R = 0 \tag{26}$$

The full solution is given by:

$$R(x) = C_1e^{-ikx}x^{l-s}U\left(1-l-s, 2l+2, 2ikx\right) + C_2e^{-ikx}x^{-l-s-1}U\left(-l-s, -2l, 2ikx\right) \tag{27}$$

where $U(a, b, x)$ is the confluent hypergeometric function and C_1, C_2 are constants of integration

4.4 Analytic Approach to Amplification Factors

We have solved the equations in the near horizon and far horizon limits. Now in order to do the matching procedure we must consider the near horizon limit of the far solution and the far horizon limit of the near solution and demand that since the function must be continuous throughout, the two limiting functions must be equal. To do this we, we can Taylor expand the solutions obtained in the relevant regions. Consider the expansion of the near solution given by (25). This solution is expanded at large x which gives:

$$R \sim A_1 \left[x^{l-s} \frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(c-a)\,\Gamma(b)} + x^{-l-s-1} \frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)} \right] \tag{28}$$

where a, b, c are the same constants mentioned in Section 4.3.1. Similarly, we expand the far solution (25) for $kx << 1$. This gives us:

$$R \sim B_1 x^{l-s} + B_2 x^{-l-s-1} \tag{29}$$

By continuity of the function we demand that these solutions must be equal in this regime - large values of the near solution and small values of the far solution. Thus, equating the coefficients of individual powers of x we get:

$$C_1 = A_1 \frac{\Gamma(1-s-2iQ)\,\Gamma(2l+1)}{\Gamma(l-s+1)\,\Gamma(l+1-2iQ)} \tag{30}$$

$$C_2 = A_1 \frac{\Gamma(1-s-2iQ)\,\Gamma(-1-2l)}{\Gamma(-l-2iQ)\,\Gamma(-l-s)} \tag{31}$$

We also have the asymptotic solution in (20) which we can compare with the far solution (27) by expanding for $r \to \infty$. By doing this, we can express I_s, R_s in (20) in terms of the C_1, C_2. This is given by:

$$I_s = \frac{1}{\omega} \left[k^{l+s+1} \frac{C_2 \,(-2i)^{l+s}\, \Gamma(-2l)}{\Gamma(-l+s)} + k^{s-l} \frac{C_1 \,(-2i)^{s-l-1}\, \Gamma(2l+2)}{\Gamma(l+s+1)} \right] \tag{32}$$

$$R_s = \omega^{-2s-1} \left[k^{l+s+1} \frac{C_2 \,(-2i)^{l+s}\, \Gamma(-2l)}{\Gamma(-l-s)} + k^{s-l} \frac{C_1 \,(-2i)^{-s-l-1}\, \Gamma(2l+2)}{\Gamma(l-s+1)} \right] \tag{33}$$

Note that in the above equations, $l \in Z^+$ but $s \in Z$, which means that at least some of the Γ functions can diverge. Manipulating these functions is discussed in Appendix B. Then, we have the amplification factor:

$$Z_{\text{slm}} = \left| \frac{R_s R_{-s}}{I_s I_{-s}} \right| - 1 \tag{34}$$

4.5 Numerical Approach to Amplification Factors

In a manner similar to the problem in Section 3, the authors of [2] have written a numerical program (which can be found here) in order to compute these amplification factors. Using this, we obtain the following plot for a massless scalar field in a slowly rotating black hole background, which is given in Figure 2.

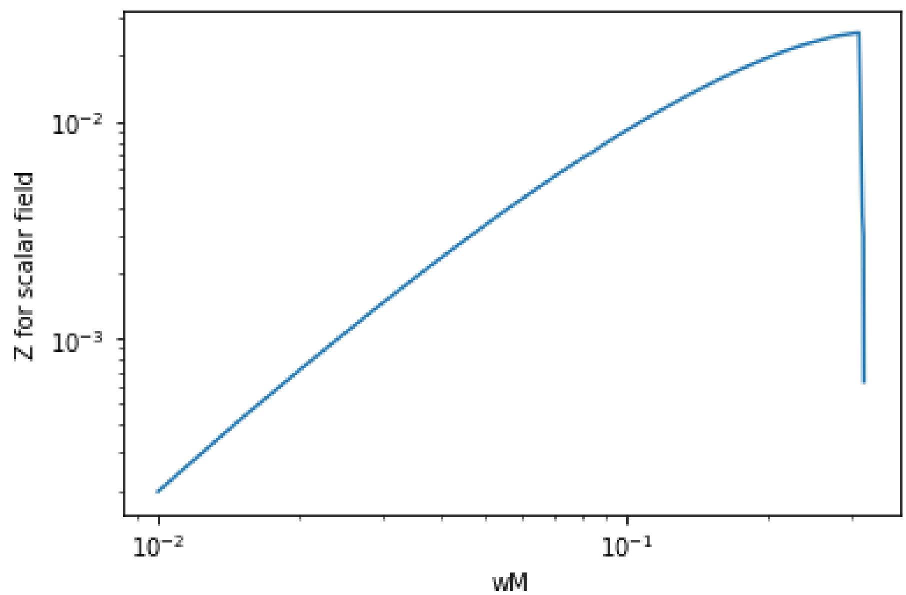

Figure 2: Z_{011} for Kerr BH with a = 0.99 M and M=1. This is obtained numerically by solving (18).

The program linked above follows a very similar approach to the program for the charged black hole case, which was discussed in Section 3.4

This concludes our discussion of superradiance in Kerr. In the following section, we will look at rotating solutions in the newly proposed Ashtekar Olmedo Singh spacetime, which are quite similar to the Kerr solution. The differences are due to quantum corrections in the spacetime.

5 Ashtekar Olmedo Singh Geometry

An extension of the classical Schwarzchild spacetime was recently proposed which includes quantum corrections inspired by loop quantum gravity. The effective line element (hereafter called AOS Black Holes) exterior to trapping and anti-trapping horizons, in static, spherically symmetric form is given by [21, 22, 23]:

$$g_{ab}dx^a dx^b = -\frac{p_b^2}{p_c\, L_o^2}dx^2 + \frac{\gamma^2 p_c\, \delta_b^2}{\sinh^2(\delta_b b)}dT^2 + p_c d\omega^2 \ ,$$

where x and T are the time and radial coordinates, respectively and $d\omega^2$ is the metric on a unit 2-sphere. The parameters appearing in metric coefficients are determined as

$$\begin{aligned}
&\tan\Big(\frac{\delta_c c\,(T)}{2}\Big) = \frac{\gamma L_0 \delta_c}{8m} e^{-2T} \ , \\
&p_c\,(T) = 4m^2\Big(e^{2T} + \frac{\gamma^2 L_0^2 \delta_c^2}{64m^2} e^{-2T}\Big) \ , \\
&\cosh\big(\delta_b\, b(T)\big) = b_o \tanh\left(\frac{1}{2}\Big(b_o T + 2\tanh^{-1}\big(\frac{1}{b_o}\big)\Big)\right) , \\
&p_b(T) = -2m\gamma L_0\, \frac{\sinh\left(\delta_b\, b(T)\right)}{\delta_b}\, \frac{1}{\gamma^2 - \frac{\sinh^2(\delta_b\, b(T))}{\delta_b^2}} \ ,
\end{aligned}$$

where m is the mass parameter and $b_o^2 = 1+\gamma^2\delta_b^2$. Here δ_b and δ_c are the quantum parameters given by,

$$\delta_b = \Big(\frac{\sqrt{\Delta}}{\sqrt{2\pi}\gamma^2 m}\Big)^{1/3} \ ; \quad L_o\delta_c = \frac{1}{2}\Big(\frac{\gamma\Delta^2}{4\pi^2 m}\Big)^{1/3} \ .$$

In the above, Δ is the minimum non-zero eigenvalue of the area operator in LQG, given by $\Delta \approx 5.17\ell_{pl}^2$ and $\gamma \approx 0.2375$ is the Barbero-Immirzi parameter. L_o is an infrared regulator in the LQG theory which makes the phase space description well-defined. The location of horizon is determined by $T = 0$. The metric can be rewritten in the usual spherically symmetric form, in the following way:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + h(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \ ,$$

in Schwarzschild like coordinates with the following change of notations that has been considered in [22]:

$$t = x, \quad r_S = 2m, \quad r = r_S\, e^T, \quad b_0 \equiv (1+\gamma^2\delta_b^2)^{\frac{1}{2}} = 1+\epsilon$$

In this case the metric coefficients are identified as [22]

$$-f(r) \ = \ -\left(\frac{r}{r_S}\right)^{2\epsilon} \frac{\left(1-\left(\frac{r_S}{r}\right)^{1+\epsilon}\right)\left(2+\epsilon+\epsilon\left(\frac{r_S}{r}\right)^{1+\epsilon}\right)^2}{16\left(1+\frac{\delta_c^2 L_0^2\gamma^2 r_S^2}{16r^4}\right)(1+\epsilon)^4}\left((2+\epsilon)^2-\epsilon^2\left(\frac{r_S}{r}\right)^{1+\epsilon}\right) \ ;$$

$$\frac{1}{g(r)} \ = \ \left(1+\frac{\delta_c^2 L_0^2\gamma^2 r_S^2}{16r^4}\right)\frac{\left(\epsilon+\left(\frac{r}{r_S}\right)^{1+\epsilon}(2+\epsilon)\right)^2}{\left(\left(\frac{r}{r_S}\right)^{1+\epsilon}-1\right)\left(\left(\frac{r}{r_S}\right)^{1+\epsilon}(2+\epsilon)^2-\epsilon^2\right)} \ ;$$

$$h(r) = 4m^2\left(e^{2T} + \frac{\gamma^2 L_0^2 \delta_c^2}{64m^2} e^{-2T}\right) = r^2\left(1 + \frac{\gamma^2 L_0^2 \delta_c^2 r_S^2}{16r^4}\right) .$$

We will work with the quantum parameters δ_c and ϵ from here on with the metric written in this more familiar form.

5.1 Rotating AOS Solutions

The quantum corrected metric so far is a non-rotating black hole solution and rotating solutions in this theory are constructed using the modified Newman Janis alogrithm [24], [16] as proposed in [25]. The line element for this Kerr-like AOS Black hole is given by:

$$ds^2 = -Fdt^2 - 2a\sin^2\theta\left(\sqrt{\frac{F}{G}} - F\right)dtd\phi + \frac{H}{g(r)h(r) + a^2}dr^2 + Hd\theta^2 + \sin^2\theta\left[H + a^2\sin^2\theta\left(2\sqrt{\frac{F}{G}} - F\right)\right]d\phi^2 \tag{35}$$

where

$$\begin{aligned}
F(r,\theta) &= \frac{g(r)h(r) + a^2\cos^2\theta}{(k(r) + a^2\cos^2\theta)^2}H(r,\theta) \\
G(r,\theta) &= \frac{g(r)h(r) + a^2\cos^2\theta}{H} \\
k(r) &= \sqrt{\frac{g(r)}{f(r)}}h(r) \\
g(r) &= \frac{\left((\frac{r}{r_s})^{1+\epsilon} - 1\right)\left((\frac{r}{r_s})^{1+\epsilon}(2+\epsilon)^2 - \epsilon^2\right)}{\left((\frac{r}{r_s})^{1+\epsilon}(2+\epsilon) + \epsilon\right)^2\left(1 + \frac{\delta_C^2 L_0^2 \gamma^2 r_s^2}{16r^4}\right)} \\
h(r) &= r^2\left(1 + \frac{\delta_C^2 L_0^2 \gamma^2 r_s^2}{16r^4}\right) \\
f(r) &= (\frac{r}{r_s})^{2\epsilon}\frac{\left(1 - \frac{r}{r_s}^{1+\epsilon}\right)\left(2 + \epsilon + \epsilon(\frac{r}{r_s})^{1+\epsilon}\right)^2}{16(1+\epsilon)^4\left(1 + \frac{\delta_C^2 L_0^2 \gamma^2 r_s^2}{16r^4}\right)} \\
H &\equiv H(r,\theta) \text{ is an undetermined function}
\end{aligned}$$

Here, ϵ, δ_C are the quantum parameters. Setting these parameters to zero should recover the Kerr black hole, which will be a consistent checking mechanism. Few important limits are:

$$\begin{aligned}
h(r) &\to r^2 \\
\sqrt{\frac{g(r)}{f(r)}} &\to 1 \\
k(r) &\to r^2
\end{aligned}$$

Motivated by these limits and keeping the Kerr case in mind, we take the following definitions to make the algebra easier:

$$\Delta(r) = g(r)h(r) + a^2 = GH + a^2 \sin^2\theta \tag{36}$$
$$\rho^2(r,\theta) = k(r) + a^2\cos^2\theta \tag{37}$$

Now, a general axisymmetric rotating black hole has a line element of the following form [26], [27]:

$$ds^2 = -\frac{N2 - W^2\sin^2\theta}{K^2}dt^2 - 2Wr\sin^2\theta dt d\phi + K^2r^2\sin^2\theta d\phi^2 + \frac{\Sigma^2}{r^2}\left(\frac{B^2}{N^2}dr^2 + r^2 d\theta^2\right) \tag{38}$$

where, N, W, K, B are functions which can be taken to be of the form:

$$\begin{aligned}
B(r,\theta) &= R_B(r) \\
\Sigma(r,\theta) &= r^2 R_\Sigma(r) + a^2\cos^2\theta \\
W(r,\theta) &= \frac{aR_M(r)}{\Sigma(r,\theta)} \\
N^2(r,\theta) &= R_\Sigma(r) - \frac{R_M(r)}{r} + \frac{a^2}{r^2} \\
K^2(r,\theta) &= \frac{1}{\Sigma(r,\theta)}\left[r^2R_\Sigma^2(r) + a^2R_\Sigma(r) + a^2\cos^2\theta N^2(r,\theta)\right] + \frac{aW(r,\theta)}{r}
\end{aligned}$$

Comparing coefficients between (38) and (35) will be useful in the following. Also, choosing $R_\Sigma = 1$ recovers the Kerr metric when $R_B = 1, R_M = 2M$ in (38).

5.2 Horizons

For the rotating AOS black hole, we do not yet have an analytical expression for the horizon. This is difficult since the roots of $\Delta = 0$ generally give the horizon, but here, the quantum parameters are found in the exponent of the variable r in the function, if one explicitly writes it down. In this context, we consider "small" quantum corrections such that one can expand Δ about the Kerr outer horizon. That is to say, we will consider a Taylor's expansion of Δ upto 2nd order about $r = r_+^{\text{Kerr}}$ and up to first order about $\epsilon = 0$, which gives us two roots for Δ. On considering the classical limit, we find that one of the roots reduces to r_+^{Kerr}. We denote this root as r_+ in what follows, and is (up to first order in ϵ) the outer horizon of the rotating AOS metric. The other root represents some other surface which does not have the interpretation of the horizon, and we shall denote this by r'. Thus, around $r = r_+^{\text{Kerr}}$, we can approximately write Δ in the form $\Delta = (r - r_+)(r - r')$, upto first order in ϵ. The actual expressions are included in a Mathematica notebook [28].

Usually, demanding that $\Delta = 0$ has real roots is used to find the location of the horizon and to avoid a naked singularity. For the Kerr black hole, this is achieved by the condition $M > a$. In the AOS case, this is modified. Given the form of Δ, one can check whether roots to this equation exist, in the region where the classical condition holds. That is, we want to check the horizon structure of the rotating AOS black hole, inside the classically allowed region. In doing so, we find the following parameter space. We see that even with maintaining $M > a$, certain values of a are only available to black holes with higher masses since Δ becomes completely positive and hence has no roots, when $\frac{a}{m}$ is high. The precise threshold of $\frac{a}{m}$ past which Δ fails to have roots depends on the value of M and is seen to increase, for increasing M. This can be observed from the parameter space plot, shown in Fig (3). The complete region represents the classically allowed region of parameters. The blue curve and the region under the curve represents the threshold at which the rotating AOS black hole horizon ceases to exist since Δ ceases to have roots beyond the threshold value of $\frac{a}{M}$. Therefore, even in the classically allowed region, the quantum corrections play a role so as to further restrict the parameter space under which the rotating AOS black hole horizon is well defined. Note that this does not require the first order expansion in ϵ, δ since this is obtained directly from the function Δ. A simulation showing the change in the function Δ is included in the Mathematica notebook attached [28].

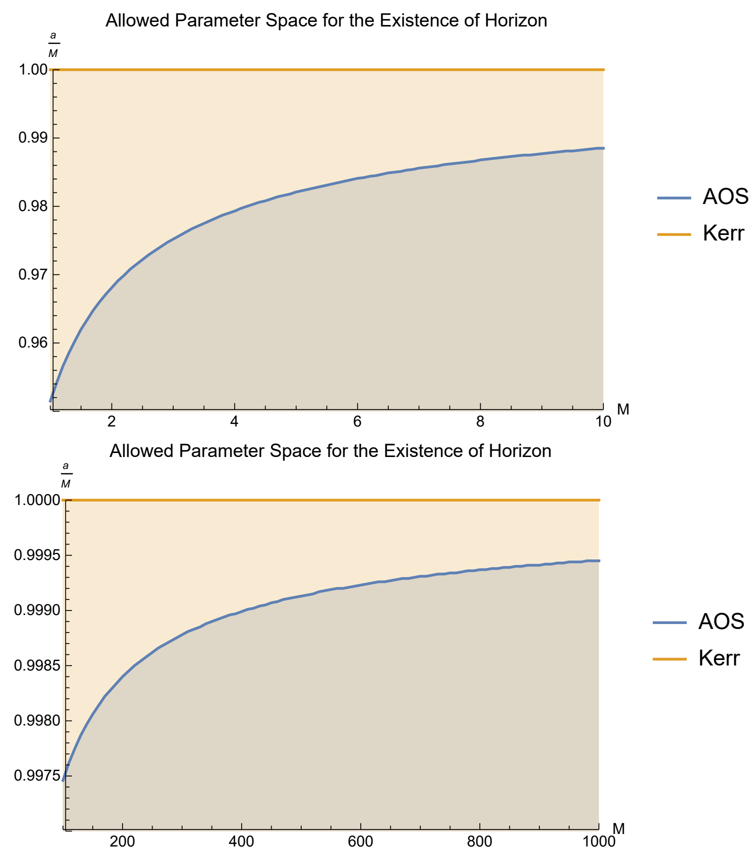

Figure 3: Comparing allowed parameter space in a/M vs M plane for Kerr and AOS.

We now follow the Teukolsky approach and stick to scalar field scattering in AOS.

5.3 Scattering of Scalar Fields in AOS Spacetime

Following the study of scattering in Kerr spacetime, one can construct an analogous transformation using the Newman Penrose formalism to look at scattering of spin fields in this rotating black hole background. However, since the associated Newman Penrose scalars for scalar fields are the fields themselves, as a first step we can directly compute the Klein Gordon equation for a massless scalar field in this background. This can be written as $\Box\psi = 0$ where:

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right) \tag{39}$$

with g being the determinant of the metric $g_{\mu\nu}$ and $g^{\mu\nu}$ being the inverse metric. Using (35) we can calculate the components $g^{\mu\nu}$. This is done using Mathematica and are rewritten in the following form:

$$\begin{aligned}
g^{tt} &= -\frac{\rho^4 + 2a^2\sin^2\theta\rho^2 + a^4\sin\theta^4 - a^2\sin^2\theta}{\Delta H} \\
g^{t\phi} &= \frac{a\Delta - a^3\sin^2\theta - a\rho^2}{\Delta H} \\
g^{rr} &= \frac{\Delta}{H} \\
g^{\theta\theta} &= \frac{1}{H} \\
g^{\phi\phi} &= \frac{\Delta - a^2\sin^2\theta}{\Delta H\sin^2\theta}
\end{aligned}$$

Expanding (39) and after some algebra, we have the equation:

$$\begin{aligned}
&\left(\frac{(k+a^2)^2 - \Delta a^2\sin^2\theta}{\Delta}\right)\frac{\partial^2\psi}{\partial t^2} - \frac{2a\,(gh-k)}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} - \left(\frac{\Delta - a^2\sin^2\theta}{\Delta\sin^2\theta}\right)\frac{\partial^2\psi}{\partial\phi^2} \\
&\qquad - \frac{\rho^2}{H}\frac{\partial}{\partial r}\left(\frac{H\Delta}{\rho^2}\frac{\partial\psi}{\partial r}\right) - \frac{\rho^2}{H\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{H\sin\theta}{\rho^2}\frac{\partial\psi}{\partial\theta}\right) = 0
\end{aligned} \tag{40}$$

Using the classical limits for the functions $k(r), \Delta(r)$ above, we see that this equation reduces to:

$$\begin{aligned}
&\left(\frac{(r^2+a^2)^2 - \tilde{\Delta}a^2\sin^2\theta}{\tilde{\Delta}}\right)\frac{\partial^2\psi}{\partial t^2} + \frac{4aMr}{\tilde{\Delta}}\frac{\partial^2\psi}{\partial t\partial\phi} + \left(\frac{a^2\sin^2\theta - \tilde{\Delta}}{\tilde{\Delta}\sin^2\theta}\right)\frac{\partial^2\psi}{\partial\phi^2} \\
&\qquad - \frac{\tilde{\rho^2}}{H}\frac{\partial}{\partial r}\left(\frac{H\tilde{\Delta}}{\tilde{\rho^2}}\frac{\partial\psi}{\partial r}\right) - \frac{\tilde{\rho^2}}{H\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{H\sin\theta}{\tilde{\rho^2}}\frac{\partial\psi}{\partial\theta}\right) = 0
\end{aligned} \tag{41}$$

where Δ, ρ^2 have reduced to $\tilde{\Delta} = r^2 - 2Mr + a^2$ and $\tilde{\rho^2} = r^2 + a^2\cos^2\theta$ which are the usual variables in the Kerr case. We see that the terms that do not depend on the function $H(r,\theta)$ are equal to the corresponding coefficients of the Klein Gordon equation in Kerr space time - as they should be, since the classical limit of the Kerr-like AOS black hole should reduce to the Kerr black hole.

Hence, for the classical limit of the equation in the AOS case to be entirely equivalent to that in the Kerr case, we can choose the undetermined function $H(r, \theta)$ as:

$$H(r, \theta) = \rho^2(r, \theta)$$

Now, using the line element for the Kerr like AOS Black Hole and the line element for a generic rotating black hole, we can compare the $g_{\theta\theta}$ components to find:

$$\Sigma^2(r, \theta) = H(r, \theta)$$

However, $\Sigma^2(r, \theta) = r^2 R_\Sigma(r) + a^2 \cos^2\theta$ and choosing $R_\Sigma = 1$ gives us Kerr again. Hence, we take take:

$$H = \rho^2 = r^2 R_\Sigma(r) + a^2 \cos^2\theta$$

as a starting point.
But $\rho^2 = k(r) + a^2 \cos^2\theta$. Using the explicit form of $k(r)$ and $h(r)$ from [25], we choose

$$R_\Sigma = \left(1 + \frac{\gamma^2 L_0^2 \delta_C^2 r_s^2}{16 r^4}\right)\sqrt{\frac{g}{f}}$$

Taking the classical limit, $\frac{\gamma^2 L_0^2 \delta_C^2 r_s^2}{16 r^4} \to 0 \implies R_\Sigma \to 1$ thus, recovering the Kerr case again.

With this choice of functional form for H the Klein Gordon equation in the Kerr-like AOS spacetime is given by:

$$\left(\frac{(k+a^2)^2 - \Delta a^2 \sin^2\theta}{\Delta}\right)\frac{\partial^2\psi}{\partial t^2} - \frac{2a\,(gh-k)}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} - \left(\frac{\Delta - a^2\sin^2\theta}{\Delta\sin^2\theta}\right)\frac{\partial^2\psi}{\partial\phi^2} - \frac{\partial}{\partial r}\left(\Delta\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) = 0 \quad (42)$$

To separate this equation we can take an ansatz for ψ of the form:

$$\psi = R(r)S(\theta)e^{-i\omega t + im\phi} \quad (43)$$

Substituting this in (42) we have the separated equations:

$$\frac{d}{dr}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \left(\frac{\omega^2(k+a^2)^2 + 2m\omega a\,(gh-k) + m^2a^2}{\Delta} - a^2\omega^2 - A_{0lm}\right)R = 0 \quad (44)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right) + \left(\omega^2a^2\cos^2\theta - \frac{m^2}{\sin^2\theta} + A_{0lm}\right)S = 0 \quad (45)$$

where $\lambda = A_{0lm} + a^2\omega^2 - 2am\omega$ is the constant of separation.

Comparing the angular equation (45) with what one would obtain in Kerr, we can see that they are identical. In fact, this is to be expected since the quantum corrections are only found in the functions of the metric that depend on r and do not have any pure angular dependence. Consequently, the solution to $S(\theta)$ is the spin weighted spheroidal harmonics (with spin $s = 0$ for scalar fields). If we consider sufficiently small $a\omega$ then the angular eigenvalues A_{0lm} can be written as:

$$A_{0lm} = l(l+1) + \mathcal{O}(a^2\omega^2) \tag{46}$$

We can now proceed to look at the radial equation.

5.4 The Radial Equation

The radial part of the Klein Gordon equation is given by:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \left(\frac{K^2}{\Delta} - \lambda\right)R = 0 \tag{47}$$

where $K = (k(r) + a^2)\omega - ma$.
To solve this, we can write it in the Schrödinger like form, as in (3). To do this, first we define the tortoise coordinate r_*:

$$\frac{\mathrm{d}r}{\mathrm{d}r_*} = \frac{\Delta}{k(r) + a^2} \equiv \mu(r) \tag{48}$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}r} = \frac{1}{\mu(r)}\frac{\mathrm{d}}{\mathrm{d}r_*}$$

Using this, and rewriting (47), we have:

$$\frac{\mathrm{d}^2R}{\mathrm{d}r_*^2} + \frac{k'\Delta}{(k+a^2)^2}\frac{\mathrm{d}R}{\mathrm{d}r_*} + \left(\frac{K^2 - \Delta(\lambda + a^2\omega^2 - 2am\omega)}{(k(r)+a^2)^2}\right)R = 0$$

Now, we take a transformation to define a new function: $\Phi = \sqrt{k+a^2}R$. Using this, we can write down the final equation as:

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}r_*^2} + V_{\text{eff}}\,\Phi = 0 \tag{49}$$

where

1. $V_{\text{eff}=}\left(\frac{K^2-\Delta(\lambda+a^2\omega^2-2am\omega)}{(k+a^2)^2} - u(r)^2 - \mu u'(r)\right)$ which is implicitly a function of r_* through (48)
2. $u(r) = \frac{\mu k'}{2(k+a^2)^2}$
3. $K = (k+a^2)\omega - ma$

From V_{eff}, one can see that as $r \to \infty$ the only term that is nonzero is $\left(\frac{K}{k+a^2}\right)^2 \to \omega^2$. Similarly, as $r \to r_+$, $\left(\frac{K}{k+a^2}\right)^2 \to (\omega - m\Omega_H)^2$, with $\Omega_H = \frac{a}{k(r_+)+a^2}$. Thus, the asymptotic solution to the Schrödinger like form are verified as long as we only consider in going wave modes at the horizon, or no reflection at the horizon. Therefore, we have (in the same notation as in the Schrödinger like equation):

$$\frac{k_H}{k_\infty} = \frac{(\omega - m\Omega_H)}{\omega^2} \tag{50}$$

Thus, we recover the same condition as Kerr, for superradiant amplification: $\omega < m\Omega_H$

In solving this equation, we will only consider the slowly rotating regime as mentioned above i.e $a\omega << 1$ and make the following change of variables.

$$x = \frac{r - r_+}{r_+ - r_-}$$

Using this (47) reduces to:

$$x^2(x+1)^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + x(x+1)(2x+1)\frac{\mathrm{d}R}{\mathrm{d}x} + \left[\tilde{k}^2x^4 - \lambda x(x+1) + \tilde{Q}^2\right]R = 0 \tag{51}$$

where

$$\begin{aligned}\tilde{Q} &= \frac{\omega - m\Omega_H}{4\pi\sigma_H} \\ 4\pi\sigma_H &= \frac{(r_+ - r_-)}{k(r_+) + a^2} \\ \tilde{k} &= \frac{1}{2}\frac{\partial^2 K}{\partial r^2}\bigg|_{r_+}\end{aligned}$$

Recall that in the Kerr limit, σ_H reduces to T_H which was the temperature of the Kerr black hole. For the AOS black hole, we do not ascribe any such interpretation at this stage. To solve this equation, we use the same 'matching procedure' as explained in Kerr (refer Appendix A for a detailed discussion).

5.4.1 The Near Solution

In the near horizon limit, (51) reduces to

$$x^2(x+1)^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + x(x+1)(2x+1)\frac{\mathrm{d}R}{\mathrm{d}x} + \left[-\lambda x(x+1) + \tilde{Q}^2\right]R = 0 \tag{52}$$

This has a solution of the form (satisfying in going boundary conditions at the horizon):

$$R(x) = A_1 x^{-i\tilde{Q}}(x+1)^{-i\tilde{Q}}F\left(-l, l+1, 1-2i\tilde{Q}, -x\right) \tag{53}$$

where F is the hypergeometric function and we have used $A_{0lm} = l(l+1)$ in the slow rotating approximation.

5.4.2 The Far Solution

In the far region limit $x >> 1$ and the equation becomes:

$$\frac{\mathrm{d}^2 R}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}R}{\mathrm{d}x} + \left[\tilde{k}^2 - \frac{\lambda}{x^2}\right] R = 0 \tag{54}$$

This has a solution of the form:

$$R(x) = C_1 e^{-i\tilde{k}x} x^l U\left(1-l, 2l+2, 2i\tilde{k}x\right) + C_2 e^{-i\tilde{k}x} x^{-l-1} U\left(-l, -2l, 2i\tilde{k}x\right) \tag{55}$$

where U is the confluent hypergeometric function. Matching the cross limits of (53,55) we find the constants C_1, C_2:

$$C_1 = A_1 \frac{\Gamma\left(1-2i\tilde{Q}\right)\Gamma\left(2l+1\right)}{\Gamma\left(l+1\right)\Gamma\left(l+1-2i\tilde{Q}\right)}$$

$$C_2 = A_1 \frac{\Gamma\left(1-2i\tilde{Q}\right)\Gamma\left(-1-2l\right)}{\Gamma\left(-l-2i\tilde{Q}\right)\Gamma\left(-l\right)}$$

Comparing the far solution with the asymptotic solution at ∞ we have the required coefficients:

$$I = \frac{1}{\omega}\left[\tilde{k}^{l+1}\frac{C_2\left(-2i\right)^l\Gamma\left(-2l\right)}{\Gamma\left(-l\right)} + \tilde{k}^{-l}\frac{C_1\left(-2i\right)^{-l-1}\Gamma\left(2l+2\right)}{\Gamma\left(l+1\right)}\right]$$

$$R = \frac{1}{\omega}\left[\tilde{k}^{l+1}\frac{C_2\left(2i\right)^l\Gamma\left(-2l\right)}{\Gamma\left(-l\right)} + \tilde{k}^{-l}\frac{C_1\left(2i\right)^{-l-1}\Gamma\left(2l+2\right)}{\Gamma\left(l+1\right)}\right]$$

Once we have obtained these coefficients, we use the definition of the amplification factor

$$Z_{lm} = \frac{\mathrm{d}E_{out}}{\mathrm{d}E_{in}} - 1 = \left|\frac{R}{I}\right|^2 - 1 \tag{56}$$

5.5 Amplification Factors and Results

5.5.1 Effect of the rotation parameter

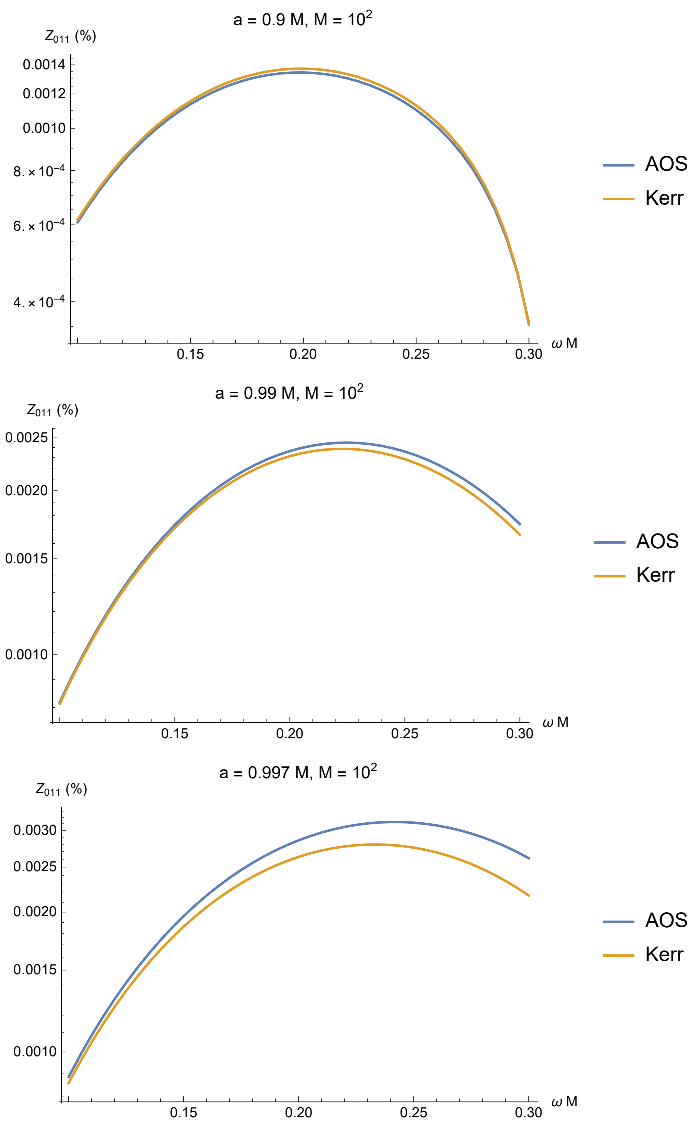

Figure 4: Comparing the effect of increasing a on Z_{011} with $M = 10^2$, We see that for black holes of a constant mass, an increase in the rotation parameter increases the separation between the AOS and Kerr superradiance. In this mass regime, superradiant amplification in AOS starts out lower than that of Kerr, but increases and exceeds Kerr with an increase in a

5.5.2 Effect of mass

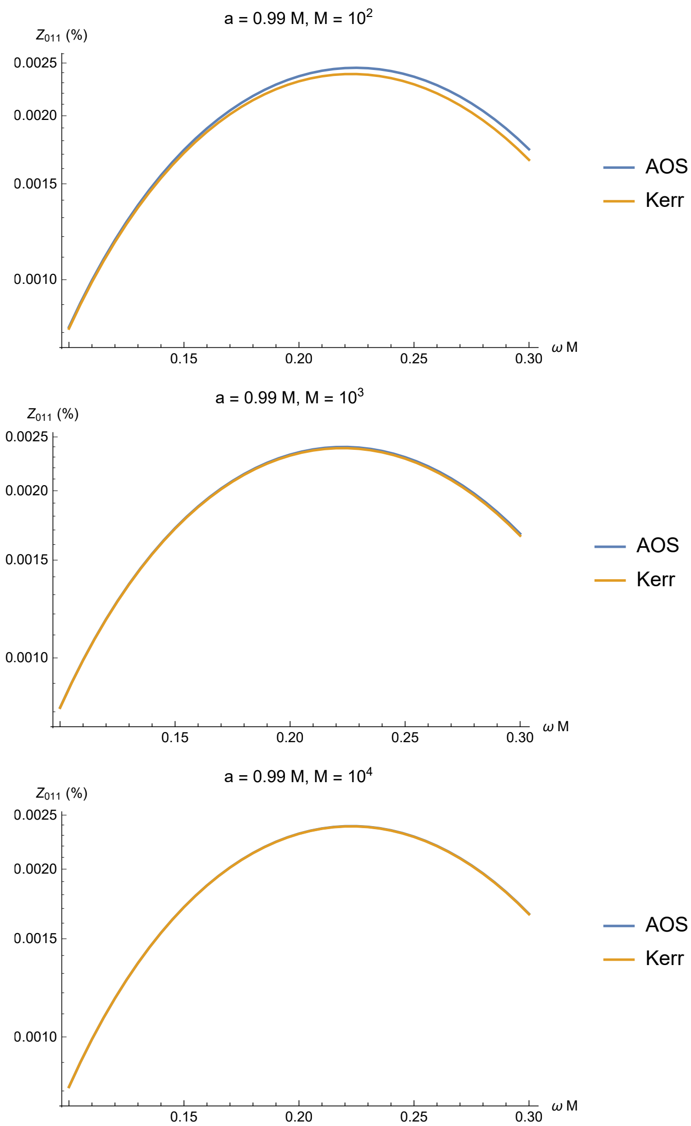

Figure 5: Comparing the effect of increasing M on Z_{11} with $a = 0.99M$.By varying the mass with a constant rotation parameter, we see the opposite effect. The amplification is much more closer between Kerr and AOS. This can be understood by noting that an increase in mass corresponds to a decrease in quantum correction effects. Thus, the large mass limit and the classical limit, are identical.

6 Discussions and Conclusions

In this thesis we have explored superradiance in the context of rotating black holes. We started with a brief introduction to superradiance in static black holes by looking at the classical Reissner-Nordström solution, then moved to the classical rotating black hole - the Kerr solution. Following a thorough review of superradiance in Kerr, we have attempted to extend this understanding to Kerr like black holes with quantum corrections proposed by AOS, inspired by loop quantum gravity.

We considered rotating solutions in this spacetime constructed using the modified Newman Janis algorithm [25], and looked at superradiance in AOS. A number of subtle complications arise, beginning with defining the new horizons, and in understanding the role of the quantum corrections in how the horizons are perturbed. It is surprising that the condition for superradiance remains the same in the AOS case, although we see that for small black holes with large rotation parameters, the amplification of AOS is higher. The quantum corrections in AOS are inversely related to the mass of the black hole. Thus, we are required to consider extremely small black holes - mass comparable to the Planck length - in order to truly work in the regime where the quantum gravity corrections dominate. This is generally not the regime in which Kerr black holes are studied, although any comparison between these two theories must happen at this length scale to truly understand the quantum effects. Although the condition for the superradiance remains the same in both cases, the allowed window of ω is larger for AOS black holes. As the mass of the black hole increases however, the quantum corrections naturally decrease and one near identical behaviour between Kerr and AOS. A natural next step would be to understand superradiant scattering of other spin fields in this background which can be done by making a straightforward generalisation of the Kinnersely tetrad in the Newman Penrose formalism to the AOS case.

A Solving the Radial Equation in Kerr

In this section, we discuss the solution of the radial equation for Kerr spacetime. The solution in AOS spacetime is identical.

A.1 Near Horizon Solution in Kerr

To solve equation (24) we can redefine the function in the following ways.

$$R(x) = \frac{U_1(x)}{x^{s+iQ}}$$

Then we have,

$$\frac{dU_1}{dx} = -(s+iQ)x^{-1-(s+iQ)}R(x) + x^-s + iQ\frac{dR}{dx}$$
$$\frac{d^2U_1}{dx^2} = -(-1-(s+iQ))(s+iQ)x^{-2-(s+iQ)}R(x) - 2(s+iQ)x^{-1-(s+iQ)}\frac{dR}{dx} + x^{-(s+iQ)}\frac{d^2R}{dx^2}$$

Using this definition, the equation simplifies to:

$$x(1+x)\frac{d^2U_1}{dx^2} - (1+x)\left[(1-s+2x-2iQ(1+x)\right]\frac{dU_1}{dx} + U_1\left[l(1+x)+l^2(1+x)+Q(Q(2+x)+I(1+s+x))\right] = 0$$

Next, we define

$$U_2(x) = \frac{U_1(x)}{(x+1)^{iQ-s}}$$

Using this, we have the equation:

$$-x(x+1)\frac{d^2U_2}{dx^2} + (2IQ + (-1+s)(1+2x))\frac{dU_2}{dx} - (s(s+1) - l(l+1))U_2 = 0$$

The standard form of the Euler Hypergeometric differential equation is given by [29]:

$$u(1-u)\frac{d^2f(u)}{du^2} + (c - (a+b+1))\frac{df(u)}{du} - abf(u) = 0$$

The above equation can be brought to this form by taking

$$\begin{aligned} u &= -x \\ a &= -(l+s) \\ b &= l - s + 1 \\ c &= 1 - s - 2iQ \end{aligned}$$

Thus, we have:

$$U_2 = F(a, b, c, -x)$$

with a, b, c defined as above and F is the hypergeometric function. Combining the functions U_1, U_2 to find the original radial function, we get the solution in (25)

A.2 Far Horizon Solution

Starting with (26) we can divide by x^4 and using $\frac{1}{x^3} << 1$, we have:

$$\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + \frac{2(1+s)}{x}\frac{\mathrm{d}R}{\mathrm{d}x} + \left[k^2 + \frac{2isk}{x} - \frac{\lambda}{x^2}\right]R = 0$$

To solve this equation, we again redefine the function appropriately. Firstly, consider:

$$\begin{aligned} P_1(x) &= e^{-ikx}R(x) \\ P_1'(x) &= e^{-ikx}\left(-ikR(x) + R'(x)\right) \\ P_1''(x) &= e^{-ikx}\left(R''(x) - k^2R(x) - 2ikR'(x)\right) \end{aligned}$$

Therefore, rewriting the ODE in terms of P_1, we have:

$$x\frac{\mathrm{d}^2P_1}{\mathrm{d}x^2} + 2x(1+s-ikx)\frac{\mathrm{d}P_1}{\mathrm{d}x} + (-l - l^2 + s + s^2 - 2ikx)P_1 = 0$$

From this, consider:

$$\begin{aligned}
P_2(x) &= x^{l-s}P_1(x)\\
P_2'(x) &= \left[(l-s)x^{l-s-1}P_1(x) + x^{l-s}P_1'(x)\right]\\
P_2''(x) &= \left[(l-s)(l-s-1)x^{l-s-2}P_1(x) + 2(l-s)x^{l-s-1}P_1'(x) + x^{l-s}P_1''(x)\right]
\end{aligned}$$

Thus, in terms of P_2 the ODE simplifies to:

$$x\frac{\mathrm{d}^2P_2}{\mathrm{d}x^2} + 2(1+l-ikx)\frac{\mathrm{d}P_2}{\mathrm{d}x} - 2ik(1+l-s)P_2 = 0$$

Now, we make another coordinate transformation such that $u = 2ikx$. Doing this gives us the final equation:

$$u\frac{\mathrm{d}^2P_2}{\mathrm{d}u^2} + \left[2(l+1) - u\right]\frac{\mathrm{d}P_2}{\mathrm{d}u} - (-l+s-1)P_2 = 0$$

This resembles the standard form of the Confluent Hypergeometric differential equation which is [29]:

$$y\frac{\mathrm{d}^2f}{\mathrm{d}y^2} + \left[b - y\right]\frac{\mathrm{d}f}{\mathrm{d}y} - af = 0$$

with

$$\begin{aligned}
y &= u = 2ikx\\
a &= -(-l+s-1)\\
b &= 2l+2
\end{aligned}$$

Therefore we have:

$$P_2(x) = U\left(l+1-s, 2l+2, 2ikx\right)$$

where $U(a,b,x)$ denotes the confluent hypergeometric function.
In a similar way, we get another linearly independent solution which is

$$P_2(x) = \frac{U\left(-l-s, -2l, 2ikx\right)}{x^(2l+1)}$$

B Manipulating Γ functions

We encounter Γ functions with negative arguments in the amplification factors of both the Kerr and the Kerr like AOS BH. To circumvent this, we can manipulate the functions using the following properties.

B.1 Method 1

From the integral representation of $\Gamma(x)$ we have, for $z \in \mathcal{C}, z \notin Z^-$

$$\Gamma(z+1) = z\Gamma(z)$$
$$\Gamma(1-z) = (-z)\Gamma(-z)$$

We use this to prove the following result for $n \in Z$.

Claim:

$$\Gamma(z-n) = (-1)^{n-1}\frac{\Gamma(-z)\Gamma(1+z)}{\Gamma(1+n-z)}$$

Proof: For $n = 1$, consider the RHS:

$$\frac{\Gamma(-z)\Gamma(1+z)}{\Gamma(2-z)} = \frac{\Gamma(-z)\Gamma(1+z)}{(1-z)(-z)\Gamma(-z)} = \frac{\Gamma(1+z)}{z(z-1)} = \Gamma(z-1) = LHS$$

If we assume this holds for $n = k$ we have:

$$\Gamma(z-k) = (-1)^{k-1}\frac{\Gamma(-z)\Gamma(1+z)}{\Gamma(1+k-z)}$$

By the method of induction, we would have to prove this for $n = k+1$.
Considering the LHS

$$\Gamma(z-k-1) = \frac{\Gamma(z-k)}{(z-k-1)} = (-1)^{k-1}\frac{\Gamma(-z)\Gamma(1+z)}{(z-k-1)\Gamma(1+k-z)} = (-1)^{k}\frac{\Gamma(-z)\Gamma(1+z)}{\Gamma(2+k-z)}$$

which is the required result.

B.2 Method 2

We use the Γ function reflection formula:

$$\frac{\Gamma(s-a+1)}{\Gamma(s-b+1)} = (-1)^{b-a}\frac{\Gamma(b-s)}{\Gamma(a-s)}$$

for $a, b \in Z$ and complex s

B.3 Method 3

A last resort is to consider l, s as 'nearly' integers as given in [30]. In these cases, we can use the following properties of the Γ function:
We have Euler's reflection formula, for $z \notin Z$ [29]:

$$\Gamma(1-z)\Gamma(z)\frac{\pi}{\sin \pi z} \tag{57}$$

Using this, one can show that:

$$\frac{\Gamma(-l-1)}{\Gamma(-l/2)} = \frac{1}{2\cos{(\pi l)}/2}\frac{\Gamma(1+l/2)}{\Gamma(l+2)}$$

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